Hamiltonian DGFEM for compressible stratified Euler equations

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The associated weak form we will be using is the Poisson bracket for the compressible stratified Euler equations in the computational domain Ω_h. We let e index for an element and S index for the trace of an element,

$$
\begin{aligned}
\{\mathcal{F},\mathcal{G}\} = &\sum_{e\in\Omega_h}\int_e -\nabla(\rho_0\frac{\delta\mathcal{G}}{\delta\rho_h})\cdot\frac{\delta\mathcal{F}}{\delta(\rho_0\mathbf{u}_h)} + \nabla(\rho_0\frac{\delta\mathcal{F}}{\delta\rho_h})\cdot\frac{\delta\mathcal{G}}{\delta(\rho_0\mathbf{u}_h)}\ \mathrm{d}e \\
&+ \sum_{\partial e\in S}\int_{\partial e}(\rho_0^-\frac{\delta\mathcal{G}}{\delta\rho_h^-} - \rho_0^+\frac{\delta\mathcal{G}}{\delta\rho_h^+})\underline{\hat{n}}^-\cdot\left[(1-\theta)\frac{\delta\mathcal{F}}{\delta\rho_0\mathbf{u}_h^-} + \theta\frac{\delta\mathcal{F}}{\delta\rho_0\mathbf{u}_h^+}\right] \\
&- (\rho_0^-\frac{\delta\mathcal{F}}{\delta\rho_h^-} - \rho_0^+\frac{\delta\mathcal{F}}{\delta\rho_h^+})\underline{\hat{n}}^-\cdot\left[(1-\theta)\frac{\delta\mathcal{G}}{\delta\rho_0\mathbf{u}_h^-} + \theta\frac{\delta\mathcal{G}}{\delta\rho_0\mathbf{u}_h^+}\right]\ \mathrm{d}S \\
&+ \sum_{e\in\Omega_h}\int_e \frac{d\rho_0}{dz}\left(\frac{\delta\mathcal{G}}{\delta\rho_h}\frac{\delta\mathcal{F}}{\delta\rho_0 w_h} - \frac{\delta\mathcal{F}}{\delta\rho_h}\frac{\delta\mathcal{G}}{\delta\rho_0 w_h}\right)\ \mathrm{d}e \\
&+ \sum_{e\in\Omega_h}\int_e \rho_0 g\left(\frac{\delta\mathcal{F}}{\delta p_h}\frac{\delta\mathcal{G}}{\delta\rho_0 w_h} - \frac{\delta\mathcal{G}}{\delta p_h}\frac{\delta\mathcal{F}}{\delta\rho_0 w_h}\right)\ \mathrm{d}e \\
&+ \sum_{e\in\Omega_h}\int_e -\nabla\big(c_0^2\rho_0\frac{\delta\mathcal{G}}{\delta p_h}\big)\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{u}_h} + \nabla\big(c_0^2\rho_0\frac{\delta\mathcal{F}}{\delta p_h}\big)\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{u}_h}\ \mathrm{d}e \\
&+ \sum_{\partial e\in S}\int_{\partial e}(c_0^{2^-}\rho_0^-\frac{\delta\mathcal{G}}{\delta p_h^-} - c_0^{2^+}\rho_0^+\frac{\delta\mathcal{G}}{\delta p_h^+})\underline{\hat{n}}\cdot\left[(1-\theta)\frac{\delta\mathcal{F}}{\delta\rho_0\mathbf{u}_h^-} + \theta\frac{\delta\mathcal{F}}{\delta\rho_0\mathbf{u}_h^+}\right] \\
&- (c_0^{2^-}\rho_0^-\frac{\delta\mathcal{F}}{\delta p_h^-} - c_0^{2^+}\rho_0^+\frac{\delta\mathcal{F}}{\delta p_h^+})\underline{\hat{n}}\cdot\left[(1-\theta)\frac{\delta\mathcal{G}}{\delta\rho_0\mathbf{u}_h^-} + \theta\frac{\delta\mathcal{G}}{\delta\rho_0\mathbf{u}_h^+}\right]\ \mathrm{d}S.
\end{aligned}
\tag{1}
$$

We consider flow in a domain with solid walls, such that our boundary conditions are $\underline{u}\cdot\underline{\hat{n}} = 0$. However to preserve skew-symmetry in the Poisson bracket we also require that there is a similar boundary condition on the test function for the velocity.

By defining the following functionals

$$
\begin{aligned}
\mathcal{F}_{(\rho_0\mathbf{u})} &= \int_\Omega \mathbf{u}(\mathbf{x},t).\mathbf{\Phi}(\mathbf{x})\ \mathrm{d}\mathbf{x}, \\
\mathcal{F}_\rho &= \int_\Omega \rho(\mathbf{x},t)\Phi(\mathbf{x})\ \mathrm{d}\mathbf{x}, \\
\mathcal{F}_p &= \int_\Omega p(\mathbf{x},t)\Xi(\mathbf{x})\ \mathrm{d}\mathbf{x},
\end{aligned}
\tag{2}
$$

where $\mathbf{\Phi}\in\mathcal{M}$ and Φ, $\Xi\in\mathcal{N}$ are arbitrary test functions, in the following function spaces

$$
\begin{aligned}
&\mathcal{M} = \{\mathbf{\Phi}\in(L^2(\Omega))^3 \text{ and } \nabla.\mathbf{\Phi}\in L^2(\Omega) : \hat{\mathbf{n}}.\mathbf{\Phi} = 0 \text{ at } \partial\Omega\}, \\
&\mathcal{N} = \{\Phi\in L^2(\Omega)\}.
\end{aligned}
\tag{3}
$$

We see that the boundary condition for the test function can be included in its function space.