# Compressible Stratified Waves

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## 1 Continuum Description

In this report we will consider in a compressible fluid in a one-dimensional domain $\Omega$ . We will assume that viscous effects and temperature effects do not influence the motion of the fluid. We assume acoustic speed of sound $c_0^2$ have been scaled such that $c_0^2 = 1$ , and that the equations of motions have been linearised around a state of rest. We assume $\rho_0 = e^{-3z}$. This leads to the linear compressible stratified Euler equations

$$\begin{aligned} \frac{\partial(\rho_0 \underline{u})}{\partial t} &= -\frac{\partial \rho}{\partial z}, \\ \frac{\partial \rho}{\partial t} &= N^2 \rho_0 w - \frac{\partial(\rho_0 \underline{u})}{\partial z}, \end{aligned} \tag{1}$$

where $N^2 = \frac{1}{\rho_0}\frac{d\rho_0}{dz}$. We will consider in this report and we will take that $\partial\Omega$ are taken to be solid walls, such that there is no normal flow through them.

Hamiltonian dynamics of compressible fluid flow governed by equations (1) is given by

$$\{\mathcal{F}, \mathcal{H}\} = \int_\Omega \frac{\delta \mathcal{H}}{\delta \rho} \rho_0 \nabla \cdot \frac{\delta \mathcal{F}}{\delta(\rho_0 \underline{u})} - \frac{\delta \mathcal{F}}{\delta \rho} \rho_0 \nabla \cdot \frac{\delta \mathcal{H}}{\delta(\rho_0 \underline{u})} \, \mathrm{d}z, \tag{2}$$

with its associated Hamiltonian energy functional

$$\mathcal{H} = \int_\Omega \frac{1}{2\rho_0} (|\rho_0 \underline{u}|^2 + \rho^2) \, \mathrm{d}z.$$

By using equation (**??**) we can show that the variational derivatives for our associated Hamiltonian are

$$\frac{\delta \mathcal{H}}{\delta \rho} = \frac{\rho}{\rho_0}, \quad \frac{\delta \mathcal{H}}{\delta(\rho_0 \underline{u})} = \frac{(\rho_0 \underline{u})}{\rho_0}. \tag{3}$$

## 2 Discrete Description

We now approximate the physical domain $\Omega$ with the computational domain $\Omega_h$, which consists of $e$ non-overlapping elements. The set of all edges in the computational domain is $\Gamma$, which consists of interior edges, $\partial e$ and edges which lie on the domain boundary $\partial\Omega$. We introduce discrete variable $\rho_h$ and $u_h$, which are approximations to their continuous counterparts. The Poisson bracket (2) now becomes discrete

$$\{\mathcal{F}, \mathcal{H}\} = \sum_e \int_e \frac{\delta \mathcal{H}}{\delta \rho_h} \rho_0 \nabla \cdot \frac{\delta \mathcal{F}}{\delta(\rho_0 \underline{u})_h} - \frac{\delta \mathcal{F}}{\delta \rho_h} \rho_0 \nabla \cdot \frac{\delta \mathcal{H}}{\delta(\rho_0 \underline{u})_h} \, \mathrm{d}e.$$

We integrate the Poisson bracket by parts and introduce a numerical flux to create a connection between neighbouring elements,

$$
\begin{aligned}
\{\mathcal{F},\mathcal{H}\} =&\sum_e \int_e -\nabla\cdot(\frac{\delta\mathcal{H}}{\delta\rho_h}\rho_0)\frac{\delta\mathcal{F}}{\delta u_h} + \nabla\cdot(\frac{\delta\mathcal{F}}{\delta\rho_h}\rho_0)\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h}\ \mathrm{d}e \\
&+\sum_e \int_{\partial e} \frac{\delta\mathcal{H}}{\delta\rho_h}\rho_0\underline{\hat{n}}\cdot\widehat{\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h}} - \frac{\delta\mathcal{F}}{\delta\rho_h}\rho_0\underline{\hat{n}}\cdot\widehat{\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h}}\ \mathrm{d}S.
\end{aligned} \tag{4}
$$

Wide hats on expressions in the boundary integrals indicate terms which will be approximated by numerical fluxes. The following numerical fluxes are chosen to approximate wide hat terms, where $-$ and $+$ indicate traces from the left and right elements connected to the face

$$
\begin{aligned}
\widehat{\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h}} &= (1-\theta)\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h^-} + \theta\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h^+}, \\
\widehat{\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h}} &= (1-\theta)\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h^-} + \theta\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h^+},
\end{aligned} \tag{5}
$$

we will emphasis here that this choice of numerical flux was made to preserve the skew-symmetry of the Poisson bracket. We note that by summing these interior boundary integrals over each element, they contribute twice to the Poisson bracket. Thus the contribution over each element can be rewritten to a summation over each interior boundary. We also now split contributions from $\Gamma$ into contributions from interior edges and boundary edges

$$
\begin{aligned}
\{\mathcal{F},\mathcal{H}\} =&\sum_e \int_e -\nabla\cdot(\frac{\delta\mathcal{H}}{\delta\rho_h}\rho_0)\frac{\delta\mathcal{F}}{\delta u_h} + \nabla\cdot(\frac{\delta\mathcal{F}}{\delta\rho_h}\rho_0)\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h}\ \mathrm{d}e \\
&+\sum_{\partial e}\int_{\partial e}\left(\frac{\delta\mathcal{H}}{\delta\rho_h^-}\rho_0^- - \frac{\delta\mathcal{H}}{\delta\rho_h^+}\rho_0^+\right)\left((1-\theta)\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h^-}+\theta\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h^+}\right) \\
&-\left(\frac{\delta\mathcal{F}}{\delta\rho_h^-}\rho_0^- - \frac{\delta\mathcal{F}}{\delta\rho_h^+}\rho_0^+\right)\left((1-\theta)\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h^-}+\theta\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h^+}\right)\ \mathrm{d}\Gamma \\
&+\sum_{\partial\Omega}\int_{\partial\Omega_h}\left(\frac{\delta\mathcal{H}}{\delta\rho_h^-}\rho_0^- - \frac{\delta\mathcal{H}}{\delta\rho_h^+}\rho_0^+\right)\underline{\hat{n}}\cdot\left(\widehat{\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h}}\right) \\
&-\left(\frac{\delta\mathcal{F}}{\delta\rho_h^-}\rho_0^- - \frac{\delta\mathcal{F}}{\delta\rho_h^+}\rho_0^+\right)\underline{\hat{n}}\cdot\left(\widehat{\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h}}\right)\ \mathrm{d}\Gamma
\end{aligned} \tag{6}
$$

At our boundary edges we have solid wall boundaries and thus we have that

$$
u(z=0,1)=0 \implies \frac{\delta\mathcal{H}}{\delta u}(z=0,1)=0 \implies \widehat{\frac{\delta\mathcal{H}}{\delta u_h}}(z=0,1)=0.
$$

However to preserve the skew symmetry of the bracket, we also require the flux on the test function $\widehat{\frac{\delta\mathcal{F}}{\delta u_h}}$ to vanish at these boundaries. Thus in our Poisson bracket we only have surface integral contributions from interior edges, and none from boundary edges. This simplifies the Poisson bracket to

$$
\begin{aligned}
\{\mathcal{F},\mathcal{H}\} =&\sum_e \int_e -\nabla\cdot(\frac{\delta\mathcal{H}}{\delta\rho_h}\rho_0)\frac{\delta\mathcal{F}}{\delta u_h} + \nabla\cdot(\frac{\delta\mathcal{F}}{\delta\rho_h}\rho_0)\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h}\ \mathrm{d}e \\
&+\sum_{\partial e}\int_{\partial e}\left(\frac{\delta\mathcal{H}}{\delta\rho_h^-}\rho_0^- - \frac{\delta\mathcal{H}}{\delta\rho_h^+}\rho_0^+\right)\underline{\hat{n}}\cdot\left((1-\theta)\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h^-}+\theta\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})_h^+}\right) \\
&-\left(\frac{\delta\mathcal{F}}{\delta\rho_h^-}\rho_0^- - \frac{\delta\mathcal{F}}{\delta\rho_h^+}\rho_0^+\right)\underline{\hat{n}}\cdot\left((1-\theta)\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h^-}+\theta\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})_h^+}\right)\ \mathrm{d}\Gamma.
\end{aligned} \tag{7}
$$

The bracket (7) is the Poisson bracket used for the Firedrake implementation. To simplify the repeated parts of the bracket in our implementation we introduce a discrete operator:

$$
DIV(\mathbf{u},p) = \sum_e\int_e \mathbf{u}\cdot\nabla p\ \mathrm{d}e + \sum_{\partial e}\int_{\partial e}\left(p^+ - p^-\right)\underline{\hat{n}}\cdot\left((1-\theta)\mathbf{u}^- + \theta\mathbf{u}^+\right)\ \mathrm{d}\Gamma.
$$

The Poisson bracket simplifies to

$$\{\mathcal{F},\mathcal{H}\} = DIV(\frac{\delta\mathcal{H}}{\delta u_h},\rho_0\frac{\delta\mathcal{F}}{\delta\rho_h}) - DIV(\frac{\delta\mathcal{F}}{\delta u_h},\rho_0\frac{\delta\mathcal{H}}{\delta\rho_h}). \tag{8}$$

# 3 Firedrake Implementation

Taking variations of the previous bracket results in the following discrete system

$$\begin{aligned}\rho_0\dot{\underline{u}} &= -DIV(\frac{\delta\mathcal{F}}{\delta(\rho_0\underline{u})},\rho_0\frac{\delta\mathcal{H}}{\delta\rho_h})\\ \dot{\rho} &= DIV(\frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u})},\rho_0\frac{\delta\mathcal{F}}{\delta\rho_h}).\end{aligned} \tag{9}$$

To solve the above system, we must relate the physical variables to their variational derivatives.

$$\frac{\delta\mathcal{H}}{\delta\rho_h} = \frac{\rho_h}{\rho_0}, \quad \frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u_h})} = \frac{(\rho_0\underline{u_h})}{\rho_0}. \tag{10}$$

We note that the discrete variational derivatives are only related to the physical variables in a weak sense, as for instance $\frac{\rho_h}{\rho_0}$ does not belong to the same finite element space. The initial variational derivatives are calculated with a projection of the initial condition, future values at time $t = n+1$ are found by augmenting the above discrete system:

$$\frac{\delta\mathcal{H}}{\delta\rho_h}^{n+1} = \frac{\rho_h^{n+1}}{\rho_0}, \quad \frac{\delta\mathcal{H}}{\delta(\rho_0\underline{u_h})}^{n+1} = \frac{(\rho_0\underline{u_h})^{n+1}}{\rho_0}. \tag{11}$$

# 4 Timestepper

## 4.1 Implicit Midpoint rule

An implicit midpoint rule is used, as it is a known property that the scheme preserves any property of the underlying ODE upto a quadratic order. This will be sufficient for our scheme to preserve its conservation of energy.

$$\begin{aligned}\dot{y} &= f(x,y),\\ \frac{y^{n+1}-y^n}{\Delta t} &= \frac{f(x^{n+1},y^{n+1})+f(x^n,y^n)}{2},\end{aligned} \tag{12}$$