Buoy Motion in Shallow Water Waves with Inequality Constraint

October 31, 2016

1 Nonlinear dynamics

The actual buoy resides at $z = h_b(x, Z(t))$ above the bottom in a certain horizontal interval from $x_p(t) < x < L$ with x_p the edge of the buoy at the water surface. A simple V-shaped buoy has the form

$$h_b(x, Z(t)) = Z(t) - H - \tan\alpha(x - L), \tag{1}$$

where $Z - H$ is the position of the keel. We artificially extend this buoy function h_b smoothly into the entire domain for all $0 < x < L$. The water line point at x_p is implicitly defined by $h(x_p, t) = h_b(x_p, Z)$.

The constrained variational principle is

$$\begin{aligned}
0 =& \delta \int_0^T \int_0^L -\rho h \partial_t \phi - \frac{1}{2}\rho h (\partial_x \phi)^2 - \frac{1}{2}\rho g h^2 - \rho g h H_0 + \rho\lambda(h - h_b + \mu^2)\,\mathrm{d}x \\
& - MZ\dot{W} - \frac{1}{2}MW^2 - MgZ\,\mathrm{d}t \qquad (2a)\\
=& \int_0^T \int_0^L -\rho\delta h\Big(\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + g(h - H_0) - \lambda\Big) - \rho\Big(h\partial_t(\delta\phi) + H(x)\partial_x\phi\partial_x(\delta\phi)\Big) \\
& + \rho\delta\lambda(h - h_b + \mu^2) + 2\rho\lambda\mu\delta\mu\,\mathrm{d}x \\
& - \delta Z\Big(M\dot{W} + Mg + \rho\int_0^L \lambda\,\mathrm{d}x\Big) - MZ\delta\dot{W} - MW\delta W\,\mathrm{d}t \qquad (2b)\\
=& \int_0^T \int_0^L -\rho\delta h\big(\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + g(h - H_0) - \lambda\big) + \rho\delta\phi\big(\partial_t h + \partial_x(H(x)\partial_x\phi)\big) \\
& + \rho\delta\lambda(h - h_b + \mu^2) + 2\rho\lambda\mu\,\delta\mu\mathrm{d}x \\
& - \delta Z\Big(M\dot{W} + Mg + \rho\int_0^L \lambda\,\mathrm{d}x\Big) + M\delta W(\dot{Z} - W)\,\mathrm{d}t. \qquad (2c)
\end{aligned}$$

The resulting equations of motion are:

$$\begin{aligned}
\delta h:&\quad \partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + g(h - H_0) - \lambda = 0 && (3a)\\
\delta\phi:&\quad \partial_t h + \partial_x(h\partial_x\phi) = 0 && (3b)\\
\delta\lambda:&\quad h - h_b + \mu^2 = 0 && (3c)\\
\delta\mu:&\quad \lambda\,\mu = 0 && (3d)\\
\delta Z:&\quad M\dot{W} + Mg + \rho\int_0^L \lambda\,\mathrm{d}x = 0 && (3e)\\
\delta W:&\quad \dot{Z} = W. && (3f)
\end{aligned}$$

The variation $\delta\mu$ yields that $\lambda\,\mu = 0$ so either:

- $\lambda = 0$ and $\mu^2 > 0$ in the part of the domain where $h_b - h = \mu^2 > 0$, or
- $\mu = 0$ with $\lambda \neq 0$ in the part of the domain under the buoy where $h = h_b$.

So by introducing the constraint $h - h_b + \mu^2 = 0$ with global Lagrange multiplier λ we have imposed the non-negative nature of $h_b - h$ both as inequality as well as an equality. (Reference found online regarding control theory.)

2 Steady State

The steady state system at rest is $\phi = 0$, $W = 0$, $Z = \bar{Z}$ and $h = H(x)$, $h_b = H_b(x, \bar{Z})$, $\lambda = \Lambda(x)$, $\mu = \bar{\mu}(x)$ satisfying

$$
\begin{aligned}
g(h - H_0) - \lambda =& 0 && \text{(4a)}\\
h - h_b(x, \bar{Z}) + \mu^2 =& 0 && \text{(4b)}\\
\lambda\, \mu =& 0 && \text{(4c)}\\
Mg + \rho \int_0^L \lambda \, \mathrm{d}x =& 0, && \text{(4d)}
\end{aligned}
$$

which holds for a reference height $\bar{Z}$ such that the displaced water of the buoy equals its mass M. This also defines the waterline point at $x = x_p = L_p$. In steady state the solution then becomes

$$
\begin{aligned}
0 < x < L_p: &\quad H(x) = H_0, \quad \Lambda(x) = 0, \quad \bar{\mu}(x) = \sqrt{H_b(x, \bar{Z}) - H_0}, && \text{(5a)}\\
L_p \le x < L: &\quad h = H_b(x, \bar{Z}), \quad \bar{\mu}(x) = 0, \quad \Lambda(x) = g(H_b(x, \bar{Z}) - H_0), && \text{(5b)}
\end{aligned}
$$

in which we extended $H_b(x)$ artificially for $x < L_p$ to equal its value at L_p.

3 Linearisation

The following linearization is applied to the variational principle after we substitute

$$
\begin{aligned}
\phi =& \phi, \quad h = H(x) + \eta, \quad h_b(x, Z) = H_b(x, \bar{Z}) + \eta_b = H_b(x, \bar{Z}) + \tilde{Z}, && \text{(6a)}\\
\lambda =& \Lambda(x) + \tilde{\lambda}, \quad \mu = \bar{\mu}(x) + \tilde{\mu}, \quad W = W, \quad Z = \bar{Z} + \tilde{Z} && \text{(6b)}
\end{aligned}
$$

into (26a), in which constant terms can be discarded, terms linear in the perturbation variables cancel after using the steady state solution, and in which only quadratic terms in the variations are kept. Hence, we find the following variational principle for the linearized equations of motion

$$
\begin{aligned}
0 =& \delta \int_0^T \int_0^L -\rho\eta\partial_t\phi - \frac{1}{2}\rho H(x)(\partial_x\phi)^2 - \frac{1}{2}\rho g\eta^2 + \rho\tilde{\lambda}(\eta - \tilde{Z} + 2\bar{\mu}\tilde{\mu}) + \rho\Lambda\tilde{\mu}^2 \, \mathrm{d}x\\
&- M\tilde{Z}\dot{W} - \frac{1}{2}MW^2 \, \mathrm{d}t && \text{(7)}\\
=& \int_0^T \int_0^L -\rho\delta\eta(\partial_t\phi + g\eta - \tilde{\lambda}) - \rho\big(\eta\partial_t(\delta\phi) + H(x)\partial_x\phi\partial_x(\delta\phi)\big)\\
&+ \rho\delta\tilde{\lambda}(\eta - \tilde{Z} + 2\bar{\mu}\tilde{\mu}) + 2\rho\delta\tilde{\mu}(\Lambda\tilde{\mu} + \bar{\mu}\tilde{\lambda}) \, \mathrm{d}x\\
&- \delta\tilde{Z}\big(M\dot{W} + \int_0^L \tilde{\lambda} \, \mathrm{d}x\big) + M\delta W\big(\dot{\tilde{Z}} - W\big) \, \mathrm{d}t && \text{(8)}
\end{aligned}
$$

after using the end-point conditions on $\delta\phi$ and δW. These variations lead to the equations of motion

$$
\begin{aligned}
\delta\eta: &\quad \partial_t\phi + g\eta - \tilde{\lambda} = 0 && \text{(9a)}\\
\delta\phi: &\quad \partial_t\eta + \partial_x\big(H(x)\partial_x\phi\big) = 0 && \text{(9b)}\\
\delta Z: &\quad M\dot{W} + \rho\int_0^L \tilde{\lambda} \, \mathrm{d}x = 0 && \text{(9c)}\\
\delta W: &\quad \dot{Z} = W, && \text{(9d)}
\end{aligned}
$$

including

$$
\begin{aligned}
\delta\tilde{\lambda}: &\quad \eta - \tilde{Z} + 2\bar{\mu}\tilde{\mu} = 0 && \text{(9e)}\\
\delta\tilde{\mu}: &\quad \bar{\mu}\tilde{\lambda} + \Lambda\tilde{\mu} = 0. && \text{(9f)}
\end{aligned}
$$

Since $\Lambda = 0$ for $0 < x < L_p$ while $\bar{\mu} > 0$, it follows from (9f) that $\tilde{\lambda} = 0$ on that interval with a free surface. Likewise, since $\bar{\mu} = 0$ while $\Lambda \neq 0$ for $L_p \le x < L$, it follows from (9f) that $\tilde{\mu} = 0$ on that interval under the buoy.

4 Numerical Implementation

4.1 Galerkin space-time discretisation

Substitute C^0 finite element Galerkin expansions

$$h(x,t) \approx h_h(x,t) = h_m(t)\varphi_m(x), \quad \phi(x,t) \approx \phi_h(x,t) = \phi_k(t)\varphi_k(x), \tag{10}$$
$$\lambda(x,t) \approx \lambda_h(x,t) = \lambda_l(t)\varphi_l(x), \quad \mu(x,t) \approx \mu_h(x,t) = \mu_l(t)\varphi_l(x), \tag{11}$$

for h, ϕ, λ and μ into the variational principle yields

$$\begin{aligned} 0 =& \delta \int_0^T -\rho M_{kl}\eta_k\dot{\phi}_l - \frac{1}{2}\rho A_{kl}\phi_k\phi_l - \frac{1}{2}\rho g M_{kl}\eta_k\eta_l \\ &+ \rho\big(M_{kl}\tilde{\lambda}_l\eta_k - Q_l\tilde{\lambda}_l\tilde{Z} + 2B_{kl}\tilde{\lambda}_l\tilde{\mu}_k + C_{kl}\tilde{\mu}_l\tilde{\mu}_k\big) - M\tilde{Z}\dot{W} - \frac{1}{2}MW^2\,\mathrm{d}t, \end{aligned} \tag{12}$$

with matrices

$$\begin{aligned} M_{kl} &= \int_0^L \varphi_k\varphi_l\,\mathrm{d}x, \quad A_{kl} = \int_0^L H(x)\partial_x\varphi_k\partial_x\varphi_l\,\mathrm{d}x, \\ B_{kl} &= \int_0^L \bar{\mu}(x)\varphi_k\varphi_l\,\mathrm{d}x, \quad C_{kl} = \int_0^L \Lambda(x)\varphi_k\varphi_l\,\mathrm{d}x. \end{aligned} \tag{13}$$

Define N time slabs $[t_n, t_n+\Delta t_n]$ for $n = 0,\dots,N$. The variational principle with constant $\Delta t_n = T/N$ becomes

$$\begin{aligned} 0 =& \delta \sum_{n=0}^{N-1} -\rho M_{kl}(\eta_k^{n+1,-} + \eta_k^{n,+})(\phi_l^{n+1/2} - \phi_l^{n,+}) - M(\tilde{Z}^{n+1,-} + \tilde{Z}^{n,+})(W^{n+1/2} - W^{n,+}) \\ &- \frac{1}{2}\Delta t\rho\Big(A_{kl}\phi_k^{n+1/2}\phi_l^{n+1/2} + gM_{kl}\frac{1}{2}(\eta_k^{n+1,-}\eta_l^{n+1,-} + \eta_k^{n,+}\eta_l^{n,+})\Big) \\ &- \frac{1}{2}\Delta t M(W^{n+1/2})^2 + \Delta t\rho C_{kl}\tilde{\mu}_k^{n+1/2}\tilde{\mu}_l^{n+1/2} \\ &+ \Delta t\rho\tilde{\lambda}_l^{n+1/2}\Big(M_{kl}\frac{1}{2}(\eta_k^{n+1,-} + \eta_k^{n,+}) - Q_l\frac{1}{2}(\tilde{Z}^{n+1,-} + \tilde{Z}^{n,+}) + 2B_{kl}\tilde{\mu}_k^{n+1/2}\Big) \\ &+ \delta \sum_{n=-1}^{N-1} -\rho M_{kl}\eta_k^{n+1,-}(\phi_l^{n+1,+} - 2\phi_l^{n+1/2} + \phi_l^{n,+}) \\ &- M\tilde{Z}^{n+1,-}(W^{n+1,+} - 2W^{n+1/2} + W^{n,+}). \end{aligned} \tag{14}$$

The resulting equations of motion are

$$\delta\eta_k^{n,+}: \quad M_{kl}\phi_l^{n+1/2} = M_{kl}\phi_l^{n,+} - \frac{1}{2}\Delta t\, gM_{kl}\eta_l^{n,+} + \frac{1}{2}\Delta t M_{kl}\tilde{\lambda}_l^{n+1/2} \tag{15a}$$
$$\delta\tilde{Z}^{n,+}: \quad MW^{n+1/2} = MW^{n,+} - \frac{1}{2}\rho\Delta t Q_l\tilde{\lambda}_l^{n+1/2} \tag{15b}$$
$$\delta\phi_l^{n+1/2}: \quad M_{kl}\eta_k^{n+1,-} = M_{kl}\eta_k^{n,+} + \Delta t\, A_{kl}\phi_k^{n+1/2} \tag{15c}$$
$$\delta W^{n+1/2}: \quad \tilde{Z}^{n+1,-} = \tilde{Z}^{n,+} + \Delta t\, W^{n+1/2} \tag{15d}$$
$$\delta\tilde{\mu}_k^{n+1/2}: \quad B_{kl}\tilde{\lambda}_l^{n+1/2} + C_{kl}\tilde{\mu}_l^{n+1/2} = 0 \tag{15e}$$
$$\delta\tilde{\lambda}_l^{n+1/2}: \quad M_{kl}\eta_k^{n+1,-} - Q_l\tilde{Z}^{n+1,-} + 2B_{kl}\tilde{\mu}_k^{n+1/2} = 0 \tag{15f}$$
$$\delta\eta_k^{n+1,-}: \quad M_{kl}\phi_l^{n+1,+} = M_{kl}\phi_l^{n+1/2} - \frac{1}{2}\Delta t gM_{kl}\eta_l^{n+1,-} + \frac{1}{2}\Delta t M_{kl}\tilde{\lambda}_l^{n+1/2} \tag{15g}$$
$$\delta\tilde{Z}^{n+1,-}: \quad MW^{n+1,+} = MW^{n+1/2} - \frac{1}{2}\rho\Delta t\, Q_l\tilde{\lambda}_l^{n+1/2} \tag{15h}$$
$$\delta W^{n,+}: \quad \tilde{Z}^{n,-} = \tilde{Z}^{n,+} \tag{15i}$$
$$\delta\phi_l^{n,+}: \quad \eta_k^{n,+} = \eta_k^{n,-}. \tag{15j}$$

The right hand side of the constraint equation (15f) is actually equivalent to $M_{kl}\eta_l^{n,+} - Q_l\tilde{Z}^{n,+}$ but this is zero since it is valid for $n = 0$, i.e. at $t = 0$ and all other subsequent times.

The constraint equations that need to be solved first are the following,

$$
\begin{aligned}
\frac{\Delta t}{2}\big(A_{kl}+\frac{\rho}{M}Q_kQ_l\big)\tilde{\lambda}_k^{n+1/2}+\frac{2}{\Delta t}B_{kl}\tilde{\mu}_k^{n+1/2} =&\frac{1}{\Delta t}(Q_l\tilde{Z}^n-M_{kl}\eta_k^n)+Q_lW^n+A_{kl}\big(\frac{\Delta t}{2}g\eta_k^n-\phi_k^n\big), &\text{(16a)}\\
B_{kl}\tilde{\lambda}_l^{n+1/2}+C_{kl}\tilde{\mu}_l^{n+1/2} =&0. &\text{(16b)}
\end{aligned}
$$

Equivalently, we can define $\tilde{\tilde{\lambda}}_k^{n+1/2}=\frac{\Delta t}{2}\tilde{\lambda}_k^{n+1/2}$ and $\tilde{B}_{kl}=\frac{2}{\Delta t}B_{kl}$, in which case the above system can be written as

$$
\begin{pmatrix} A_{kl}+\frac{\rho}{M}Q_kQ_l & \tilde{B}_{kl}\\ \tilde{B}_{kl} & C_{kl}\end{pmatrix}\begin{pmatrix}\tilde{\tilde{\lambda}}_k^{n+1/2}\\ \tilde{\mu}_k^{n+1/2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\Delta t}(Q_l\tilde{Z}^n-M_{kl}\eta_k^n)+Q_lW^n+A_{kl}\big(\frac{\Delta t}{2}g\eta_k^n-\phi_k^n\big)\\ 0\end{pmatrix}. \tag{17}
$$

The matrix above is symmetric. Upon solving the above linear system and knowing $\tilde{\tilde{\lambda}}_k^{n+1/2}$, the rest of the equations can be solved in the following order:

$$
\begin{aligned}
M_{kl}\phi_l^{n+1/2} &= M_{kl}\phi_l^n-\frac{1}{2}\Delta t\, gM_{kl}\eta_l^n+M_{kl}\tilde{\tilde{\lambda}}_l^{n+1/2} &\text{(18a)}\\
W^{n+1/2} &= W^n-\frac{\rho}{M}Q_l\tilde{\tilde{\lambda}}_l^{n+1/2} &\text{(18b)}\\
M_{kl}\eta_k^{n+1} &= M_{kl}\eta_k^n+\Delta t\, A_{kl}\phi_k^{n+1/2} &\text{(18c)}\\
\tilde{Z}^{n+1} &= \tilde{Z}^n+\Delta t\, W^{n+1/2} &\text{(18d)}\\
M_{kl}\phi_l^{n+1} &= M_{kl}\phi_l^{n+1/2}-\frac{1}{2}\Delta t\, gM_{kl}\eta_l^{n+1}+M_{kl}\tilde{\tilde{\lambda}}_l^{n+1/2} &\text{(18e)}\\
W^{n+1} &= W^{n+1/2}-\frac{\rho}{M}Q_l\tilde{\tilde{\lambda}}_l^{n+1/2}. &\text{(18f)}
\end{aligned}
$$

Alternative formulation

Alternatively, one can exchange the expansions for ϕ and η, W and $\tilde{Z}$, as well as $\tilde{\lambda}$ and $\tilde{\mu}$. (Note that in the original formulation, $\tilde{\mu}$ was expanded wrt $\tilde{\mu}_k^{n+1,-}$ and $\tilde{\mu}_k^{n,+}$, but these two ended up being the same. Here we will expand using $\tilde{\mu}_k^{n+1/2}$). The space-time discrete variational principle in this case becomes

$$
\begin{aligned}
0 =&\delta\sum_{n=0}^{N-1}-\rho M_{kl}\eta_k^{n+1/2}(\phi_l^{n+1,-}-\phi_l^{n,+})-M\tilde{Z}^{n+1/2}(W^{n+1,-}-W^{n,+})\\
&-\frac{1}{2}\Delta t\rho\big(A_{kl}\frac{1}{2}(\phi_k^{n+1,-}\phi_l^{n+1,-}+\phi_k^{n,+}\phi_l^{n,+})+gM_{kl}\eta_k^{n+1/2}\eta_l^{n+1/2}\big)\\
&-\frac{1}{4}\Delta tM(W^{n+1,-}W^{n+1,-}+W^{n,+}W^{n,+})+\Delta t\rho C_{kl}\tilde{\mu}_k^{n+1/2}\tilde{\mu}_l^{n+1/2}\\
&+\frac{1}{2}\Delta t\rho(\tilde{\lambda}_l^{n+1,-}+\tilde{\lambda}_l^{n,+})\big(M_{kl}\eta_k^{n+1/2}-Q_l\tilde{Z}^{n+1/2}+2B_{kl}\tilde{\mu}_k^{n+1/2}\big)\\
&+\delta\sum_{n=-1}^{N-1}-\rho M_{kl}\eta_k^{n+1,+}(\phi_l^{n+1,+}-\phi_l^{n+1,-})-M\tilde{Z}^{n+1,+}(W^{n+1,+}-W^{n+1,-}).
\end{aligned}\tag{19}
$$

The resulting equations of motion are

$$
\begin{aligned}
\delta\phi_l^{n,+}:&\quad M_{kl}\eta_k^{n+1/2} = M_{kl}\eta_k^{n,+} + \frac{1}{2}\Delta t\, A_{kl}\phi_k^{n,+} &\text{(20a)}\\
\delta W^{n,+}:&\quad \tilde{Z}^{n+1/2} = \tilde{Z}^{n,+} + \frac{1}{2}\Delta t\, W^{n,+} &\text{(20b)}\\
\delta\eta_k^{n+1/2}:&\quad M_{kl}\phi_l^{n+1,-} = M_{kl}\phi_l^{n,+} - \Delta t\, g M_{kl}\eta_l^{n+1/2} + \Delta t M_{kl}\frac{1}{2}(\tilde{\lambda}_l^{n+1,-} + \tilde{\lambda}_l^{n,+}) &\text{(20c)}\\
\delta\tilde{Z}^{n+1/2}:&\quad MW^{n+1,-} = MW^{n,+} - \rho\Delta t Q_l\frac{1}{2}(\tilde{\lambda}_l^{n+1,-} + \tilde{\lambda}_l^{n,+}) &\text{(20d)}\\
\delta\tilde{\mu}_k^{n+1/2}:&\quad B_{kl}\frac{1}{2}(\tilde{\lambda}_l^{n+1,-} + \tilde{\lambda}_l^{n,+}) + C_{kl}\tilde{\mu}_l^{n+1/2} = 0 &\text{(20e)}\\
\delta\tilde{\lambda}_l^{n,+}:&\quad M_{kl}\eta_k^{n+1/2} - Q_l\tilde{Z}^{n+1/2} + 2B_{kl}\tilde{\mu}_k^{n+1/2} = 0 &\text{(20f)}\\
\delta\tilde{\lambda}_l^{n+1,-}:&\quad M_{kl}\eta_k^{n+1/2} - Q_l\tilde{Z}^{n+1/2} + 2B_{kl}\tilde{\mu}_k^{n+1/2} = 0 &\text{(20g)}\\
\delta\phi_l^{n+1,-}:&\quad M_{kl}\eta_k^{n+1,-} = M_{kl}\eta_k^{n+1/2} + \frac{1}{2}\Delta t\, A_{kl}\phi_k^{n+1,-} &\text{(20h)}\\
\delta W^{n+1,-}:&\quad \tilde{Z}^{n+1,-} = \tilde{Z}^{n+1/2} + \frac{1}{2}\Delta t\, W^{n+1,-} &\text{(20i)}\\
\delta Z^{n+1/2}:&\quad W^{n,-} = W^{n,+} &\text{(20j)}\\
\delta\eta_k^{n,+}:&\quad \phi_k^{n,+} = \phi_k^{n,-}. &\text{(20k)}
\end{aligned}
$$

In this case, we get two equations (20f), (20g) from the $\delta\tilde{\lambda}_l$ variations which are the same. Also, substituting the equations for $\eta_k^{n+1/2}$ and $\tilde{Z}^{n+1/2}$ from the first two equations into (20f), does not result in a linear equation for the Lagrange multiplier.

4.2 Firedrake implementation of linear system

$$
\begin{aligned}
\int_0^L \delta\eta(\phi_h^{n+1/2} - \tilde{\lambda}_h^{n+1/2})\,\mathrm{d}x &= \int_0^L \delta\eta(\phi_h^n - \frac{\Delta t}{2}g\eta_h^n)\,\mathrm{d}x, &\text{(21a)}\\
\int_0^L (\delta\phi_h\eta_h^{n+1} - \Delta t H(x)\partial_x(\delta\phi_h)\partial_x(\phi_h^{n+1/2}))\,\mathrm{d}x &= \int_0^L \delta\phi_h\eta_h^n\,\mathrm{d}x, &\text{(21b)}\\
\int_0^L \delta\lambda_h(\frac{1}{\Delta t}\eta^{n+1} + \frac{\rho}{M}\tilde{I} + \tilde{\bar{\mu}}\tilde{\mu}_h^{n+1/2})\,\mathrm{d}x &= \int_0^L \delta\lambda_h(\frac{1}{\Delta t}\tilde{Z}^n + W^n)\,\mathrm{d}x, &\text{(21c)}\\
\int_0^L \delta\mu(\tilde{\bar{\mu}}\tilde{\lambda}_h^{n+1/2} + \Lambda\tilde{\mu}_h^{n+1/2})\,\mathrm{d}x &= 0, &\text{(21d)}\\
\int_0^L v(\frac{\tilde{I}}{L} - \tilde{\lambda}_h^{n+1/2})\,\mathrm{d}x &= 0, &\text{(21e)}\\
W^{n+1/2} &= W^n - \frac{\rho}{M}\int_0^L \tilde{\lambda}_h^{n+1/2}\,\mathrm{d}x, &\text{(21f)}\\
\tilde{Z}^{n+1} &= \tilde{Z}^n + \Delta t W^{n+1/2}, &\text{(21g)}\\
\int_0^L \delta\eta_h\phi_h^{n+1}\,\mathrm{d}x &= \int_0^L \delta\eta_h(\phi_h^{n+1/2} - \frac{\Delta t}{2}g\eta_h^{n+1} + \tilde{\lambda}_h^{n+1/2})\,\mathrm{d}x, &\text{(21h)}\\
W^{n+1} &= W^{n+1/2} - \frac{\rho}{M}\int_0^L \tilde{\lambda}_h^{n+1/2}\,\mathrm{d}x, &\text{(21i)}
\end{aligned}
$$

where we have introduced $\tilde{\bar{\mu}} = \frac{2}{\Delta t}\bar{\mu}$, $\tilde{I} = \int_0^L \tilde{\lambda}_h^{n+1/2}\,\mathrm{d}x$ and a new test function v. Note that in equation (21c), the updates for $\tilde{Z}^{n+1}$ and subsequently $W^{n+1/2}$ are substituted from (21f)-(21g) to eliminate those scalar variables. The first five equations (21a)-(21e) need to be solved together for the five unknowns $\phi_h^{n+1/2}$, η_h^{n+1}, $\tilde{\lambda}_h^{n+1/2}$, $\tilde{\mu}_h^{n+1/2}$, $\tilde{I}$. The remaining equations (21f)-(21i) can be then solved in the order provided.

4.2.1 Preconditioning

The matrix A defining the operator of the LHS in the system of equations (21a)-(21e), i.e. multiplying the vector of unknowns $(\phi_h^{n+1/2}, \eta_h^{n+1}, \tilde{\tilde{\lambda}}_h^{n+1/2}, \tilde{\mu}_h^{n+1/2}, \tilde{I})^T$, is the following

$$\mathcal{A} = \begin{pmatrix} I & 0 & -I & 0 & 0 \\ \Delta t\,\nabla\cdot(H(x)\nabla) & I & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta t}I & 0 & \bar{\mu}I & \frac{\rho}{M}I \\ 0 & 0 & \bar{\mu}I & \Lambda I & 0 \\ 0 & 0 & -I & 0 & \frac{1}{L}I \end{pmatrix}, \tag{22}$$

where I here denotes the identity operator. The corresponding matrix for the discrete system (20) is

$$\mathcal{A}_{discrete} = \begin{pmatrix} M_{kl} & 0 & -M_{kl} & 0 & 0 \\ -\Delta t\,A_{kl} & M_{kl} & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta t}M_{kl} & 0 & \tilde{B}_{kl} & \frac{\rho}{M}Q_l \\ 0 & 0 & \tilde{B}_{kl} & C_{kl} & 0 \\ 0 & 0 & -Q_l & 0 & Q_l \end{pmatrix}. \tag{23}$$

The goal is to reduce the above matrix into a diagonal matrix with some operator for $\phi_h^{n+1/2}$ and only mass matrices for the remaining variables. Eliminating $\tilde{I}$, then $\tilde{\mu}$, $\tilde{\tilde{\lambda}}$ and η by performing appropriate manipulations in the "continuum" matrix $\mathcal{A}$, results in the (diagonal) preconditioning matrix

$$\mathcal{P} = \begin{pmatrix} I-\nabla\cdot(H(x)\nabla) & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & (\frac{\rho}{M}-\bar{\mu}^2)I & 0 & 0 \\ 0 & 0 & 0 & \Lambda I & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{L}I \end{pmatrix}. \tag{24}$$

Another suitable preconditioner could be (?)

$$\mathcal{P}_2 = \begin{pmatrix} (\frac{\rho}{M}L\Lambda-\bar{\mu}^2)I-\Lambda\nabla\cdot(H(x)\nabla) & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}. \tag{25}$$

5 Semi-linearisation in water part only

Here we choose to linearise the water part only, by writing $h = H(x)+\eta$, but we allow the domain and the buoy to remain nonlinear. The variational principle becomes

$$\begin{aligned}
0 =& \delta\int_0^T \int_0^L -\rho(H(x)+\eta)\partial_t\phi - \frac{1}{2}\rho H(x)(\partial_x\phi)^2 - \frac{1}{2}\rho g(H(x)+\eta)^2 - \rho g(H(x)+\eta)H_0 \\
&+ \rho\lambda(H(x)+\eta-h_b+\mu^2)\,\mathrm{d}x \\
&- MZ\dot{W} - \frac{1}{2}MW^2 - MgZ\,\mathrm{d}t && \text{(26a)}\\
=& \int_0^T \int_0^L -\rho\delta\eta\Big(\partial_t\phi + g\eta + g(H(x)-H_0)-\lambda\Big) - \rho\Big((H(x)+\eta)\partial_t(\delta\phi) + H(x)\partial_x\phi\partial_x(\delta\phi)\Big) \\
&+ \rho\delta\lambda(H(x)+\eta-h_b+\mu^2) + 2\rho\lambda\mu\delta\mu\,\mathrm{d}x \\
&- \delta Z\Big(M\dot{W} + Mg + \rho\int_0^L \lambda\,\mathrm{d}x\Big) - MZ\delta\dot{W} - MW\delta W\,\mathrm{d}t && \text{(26b)}\\
=& \int_0^T \int_0^L -\rho\delta\eta\big(\partial_t\phi + g\eta + g(H(x)-H_0)-\lambda\big) + \rho\delta\phi\big(\partial_t\eta + \partial_x(H(x)\partial_x\phi)\big) \\
&+ \rho\delta\lambda(H(x)+\eta-h_b+\mu^2) + 2\rho\lambda\mu\,\delta\mu\mathrm{d}x \\
&- \delta Z\Big(M\dot{W} + Mg + \rho\int_0^L \lambda\,\mathrm{d}x\Big) + M\delta W(\dot{Z}-W)\,\mathrm{d}t. && \text{(26c)}
\end{aligned}$$

The resulting equations of motion are:

$$\begin{aligned}
\delta\eta: &\quad \partial_t\phi + g\eta + g(H(x) - H_0) - \lambda = 0 && (27a)\\
\delta\phi: &\quad \partial_t\eta + \partial_x(H(x)\partial_x\phi) = 0 && (27b)\\
\delta\lambda: &\quad h - h_b + \mu^2 = 0 && (27c)\\
\delta\mu: &\quad \lambda\,\mu = 0 && (27d)\\
\delta Z: &\quad M\dot{W} + Mg + \rho\int_0^L \lambda\,\mathrm{d}x = 0 && (27e)\\
\delta W: &\quad \dot{Z} = W. && (27f)
\end{aligned}$$

The variation $\delta\mu$ implies that either:

- $\lambda = 0$ and $\mu^2 = h_b - h > 0$ in the water part of the domain, or
- $\mu = 0$ with $\lambda \neq 0$ in the part of the domain under the buoy, where $h = h_b$ is satisfied.

5.1 Firedrake implementation of semi-linear system

$$\int_0^L \delta\eta\big(\phi_h^{n+1/2} - \phi_h^n + \frac{\Delta t}{2}g\eta_h^n + \frac{\Delta t}{2}g(H(x) - H_0) - \frac{\Delta t}{2}\lambda_h^{n+1/2}\big)\,\mathrm{d}x = 0, \tag{28a}$$

$$MW^{n+1/2} - MW^n + \frac{\Delta t}{2}Mg + \rho\frac{\Delta t}{2}\int_0^L \lambda_h^{n+1/2}\,\mathrm{d}x = 0, \tag{28b}$$

$$\int_0^L \delta\phi_h(\eta_h^{n+1} - \eta_h^n) - \Delta t H(x)\partial_x(\delta\phi_h)\partial_x{\phi_h}^{n+1/2}\,\mathrm{d}x = 0 \tag{28c}$$

$$\tilde{Z}^{n+1} - \tilde{Z}^n - \Delta t W^{n+1/2} = 0 \tag{28d}$$

$$\int_0^L \delta\lambda_h\big(H(x) + \eta^{n+1} - h_b^{n+1} + \mu_h^{2^{n+1/2}}\big)\,\mathrm{d}x = 0, \tag{28e}$$

$$\int_0^L \delta\mu(\mu_h^{n+1/2}\lambda_h^{n+1/2})\,\mathrm{d}x = 0 \tag{28f}$$

$$\int_0^L \delta\eta_h\big(\phi_h^{n+1} - \phi_h^{n+1/2} + \frac{\Delta t}{2}g\eta_h^{n+1} + \frac{\Delta t}{2}g(H(x) - H_0) - \frac{\Delta t}{2}\lambda_h^{n+1/2}\big)\,\mathrm{d}x = 0, \tag{28g}$$

$$MW^{n+1} - MW^{n+1/2} + \frac{\Delta t}{2}Mg + \rho\frac{\Delta t}{2}\int_0^L \lambda_h^{n+1/2}\,\mathrm{d}x = 0. \tag{28h}$$

6 Extension to 2D ship

The steady state the solution is now given by

$$\begin{aligned}
0 < x < L_p \;\&\; L_s < x < L: &\quad H(x) = H_0, \quad \Lambda(x) = 0, \quad \bar{\mu}(x) = \sqrt{H_s(x) - H_0}, && (29a)\\
L_p \le x < L_s: &\quad h = H_s(x), \quad \bar{\mu}(x) = 0, \quad \Lambda(x) = g(H_s(x) - H_0), && (29b)
\end{aligned}$$

with $H_s(x) = H_s(x;\bar{X},\bar{Z},\psi_0) = \bar{Z} - H + \tan(\alpha \pm \psi_0)|x - \bar{X}| = \bar{Z} - H + \tan\alpha|x - \bar{X}|$. Note that we extend $H_s(x)$ artificially for $x < L_p$ and $x > L_s$ to equal its value at L_p or L_s, respectively.

The variational principle becomes

$$\begin{aligned}
0 =& \delta\int_0^T\int_0^L -\rho\eta\partial_t\phi - \frac{1}{2}\rho H(x)(\partial_x\phi)^2 - \frac{1}{2}\rho g\eta^2 + \rho\tilde{\lambda}\big(\eta - \frac{\partial H_s}{\partial\bar{X}}\tilde{X} - \frac{\partial H_s}{\partial\bar{Z}}\tilde{Z} - \frac{\partial H_s}{\partial\psi_0}\tilde{\psi} + 2\bar{\mu}\tilde{\mu}\big) + \rho\Lambda\tilde{\mu}^2\,\mathrm{d}x\\
&- M\left(\tilde{X}\dot{U} + \tilde{Z}\dot{W}\right) - \tilde{\psi}\dot{p}_\psi - \frac{1}{2}M\left(U^2 + W^2\right) - \frac{1}{2}\frac{p_\psi^2}{I_x}\,\mathrm{d}t
\end{aligned} \tag{30}$$

and therein variations yield the following evolution equations

$$\begin{aligned}
\delta\eta:&\quad \partial_t\phi + g\eta - \tilde{\lambda} = 0, && (31a)\\
\delta\phi:&\quad \partial_t\eta + \partial_x\big(H_s(x)\partial_x\phi\big) = 0, && (31b)\\
\delta\tilde{X}:&\quad \dot{U} - \tan\alpha\frac{\rho}{M}\int_0^L \operatorname{sign}(x-\bar{X})\tilde{\lambda}\,\mathrm{d}x = 0 && (31c)\\
\delta\tilde{Z}:&\quad \dot{W} + \frac{\rho}{M}\int_0^L \tilde{\lambda}\,\mathrm{d}x = 0, && (31d)\\
\delta\psi:&\quad \dot{p}_\psi - \rho\sec^2\alpha\int_0^L (x-\bar{X})\tilde{\lambda}\,\mathrm{d}x = 0, && (31e)\\
\delta U:&\quad \dot{\tilde{X}} = U, && (31f)\\
\delta W:&\quad \dot{\tilde{Z}} = W, && (31g)\\
\delta p_\psi:&\quad \dot{\tilde{\psi}} = \frac{p_\psi}{I_x}, && (31h)
\end{aligned}$$

including

$$\begin{aligned}
\delta\tilde{\lambda}:&\quad \eta - \frac{\partial H_s}{\partial\bar{X}}\tilde{X} - \frac{\partial H_s}{\partial\bar{Z}}\tilde{Z} - \frac{\partial H_s}{\partial\psi_0}\tilde{\psi} + 2\bar{\mu}\tilde{\mu} = 0 && (31i)\\
\delta\tilde{\mu}:&\quad \bar{\mu}\tilde{\lambda} + \Lambda\tilde{\mu} = 0. && (31j)
\end{aligned}$$

Since $\Lambda = 0$ for $0 < x < L_p$, $L_s < x < L$ while $\bar{\mu} > 0$, it follows from (31j) that $\tilde{\lambda} = 0$ on that interval with a free surface. Likewise, since $\bar{\mu} = 0$ while $\Lambda \neq 0$ for $L_p \le x < L_s$, it follows from (31j) that $\tilde{\mu} = 0$ on that interval under the ship.

The discretised version of the constraint system is

$$\begin{pmatrix} A_{kl} + \frac{\rho}{M}Q_kQ_l + \frac{\rho}{M}(\tan\alpha)^2 Q_k^X Q_l^X + \frac{\rho}{M}(\sec^2\alpha)^2 Q_k^\psi Q_l^\psi & \tilde{B}_{kl} \\ \tilde{B}_{kl} & C_{kl} \end{pmatrix} \begin{pmatrix} \tilde{\tilde{\lambda}}_k^{n+1/2} \\ \tilde{\mu}_k^{n+1/2} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{\Delta t}(Q_l\tilde{Z}^n - \tan\alpha\, Q_l^X\tilde{X}^n - \sec^2\alpha\, Q_l^\psi\tilde{\psi}^n - M_{kl}\eta_k^n) + \big(Q_lW^n - \tan\alpha\, Q_l^X U^n - \sec^2\alpha\, Q_l^\psi\frac{p_\psi^n}{I_x}\big) + A_{kl}\big(\frac{\Delta t}{2}g\eta_k^n - \phi_k^n\big) \\ 0 \end{pmatrix}. \tag{32}$$

with $\tilde{\tilde{\lambda}}_k^{n+1/2} = \frac{\Delta t}{2}\tilde{\lambda}_k^{n+1/2}$ and $\tilde{B}_{kl} = \frac{2}{\Delta t}B_{kl}$. The above matrix is again symmetric. Upon solving the linear system and knowing $\tilde{\tilde{\lambda}}_k^{n+1/2}$, the rest of the equations can be solved in the following order:

$$\begin{aligned}
M_{kl}\phi_l^{n+1/2} &= M_{kl}\phi_l^n - \frac{1}{2}\Delta t\, gM_{kl}\eta_l^n + M_{kl}\tilde{\tilde{\lambda}}_l^{n+1/2} && (33a)\\
U^{n+1/2} &= U^n + \frac{\rho}{M}\tan\alpha\, Q_l^X\tilde{\tilde{\lambda}}_l^{n+1/2} && (33b)\\
W^{n+1/2} &= W^n - \frac{\rho}{M}Q_l\tilde{\tilde{\lambda}}_l^{n+1/2} && (33c)\\
p_\psi^{n+1/2} &= p_\psi^n + \rho\sec^2\alpha\, Q_l^\psi\tilde{\tilde{\lambda}}_l^{n+1/2} && (33d)\\
M_{kl}\eta_k^{n+1} &= M_{kl}\eta_k^n + \Delta t\, A_{kl}\phi_k^{n+1/2} && (33e)\\
\tilde{X}^{n+1} &= \tilde{X}^n + \Delta t\, U^{n+1/2} && (33f)\\
\tilde{Z}^{n+1} &= \tilde{Z}^n + \Delta t\, W^{n+1/2} && (33g)\\
\tilde{\psi}^{n+1} &= \tilde{\psi}^n + \frac{\Delta t}{I_x}p_\psi^{n+1/2} && (33h)\\
M_{kl}\phi_l^{n+1} &= M_{kl}\phi_l^{n+1/2} - \frac{1}{2}\Delta t\, gM_{kl}\eta_l^{n+1} + M_{kl}\tilde{\tilde{\lambda}}_l^{n+1/2} && (33i)\\
U^{n+1} &= U^{n+1/2} + \frac{\rho}{M}\tan\alpha\, Q_l^X\tilde{\tilde{\lambda}}_l^{n+1/2} && (33j)\\
W^{n+1} &= W^{n+1/2} - \frac{\rho}{M}Q_l\tilde{\tilde{\lambda}}_l^{n+1/2} && (33k)\\
p_\psi^{n+1} &= p_\psi^{n+1/2} + \rho\sec^2\alpha\, Q_l^\psi\tilde{\tilde{\lambda}}_l^{n+1/2}. && (33l)
\end{aligned}$$