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Aim: solve the dimensionless incompressible Navier-Stokes

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \boldsymbol{v} = -\nabla p + \frac{1}{Re} \nabla \cdot \left(2\boldsymbol{D} - \frac{2}{3} (\nabla \cdot \boldsymbol{v})\, \boldsymbol{I} \right) \tag{1a}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1b}$$

on a square domain $(0,1) \times (0,1)$ with a symmetry axis on $x = 0$, inlet on $y = 1$, outlet $y = 0$, wall $x = 1$

$$\boldsymbol{D} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{2}$$

[**REMARK:** I know that due to incompressibility the viscous stress term on the right hand side reduces to $\frac{1}{Re} \nabla^2 \boldsymbol{v}$ but I want to keep it as it is right know, as I will further build on that system]

1.1 Weak formulation

$$\int_\Omega \boldsymbol{\chi} \cdot \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \boldsymbol{v} \right) + \boldsymbol{\chi} \cdot \nabla p - \frac{1}{Re} \chi_i \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{2}{3Re} \chi_i \frac{\partial}{\partial x_j} \left((\nabla \cdot \boldsymbol{v}) \delta_{ij} \right) \mathrm{d}\Omega = 0 \tag{3a}$$

$$\int_\Omega (\nabla \cdot \boldsymbol{v})\, \theta \, \mathrm{d}\Omega = 0 \tag{3b}$$

Integrating each term in the NS, by parts

$$\begin{aligned}
\int_\Omega \boldsymbol{\chi} \cdot \nabla p \, \mathrm{d}\Omega &= \int_{\partial\Omega} p\, \boldsymbol{\chi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma - \int_\Omega p \nabla \cdot \boldsymbol{\chi} \, \mathrm{d}\Omega \\
&= \int_C p\, \boldsymbol{\chi} \cdot \boldsymbol{n} \, \mathrm{d}s_{inlet} + \int_C p\, \boldsymbol{\chi} \cdot \boldsymbol{n} \, \mathrm{d}s_{outlet} + \int_C p\, \boldsymbol{\chi} \cdot \boldsymbol{n} \, \mathrm{d}s_{sym} + \underbrace{\int_C p\, \boldsymbol{\chi} \cdot \boldsymbol{n} \, \mathrm{d}s_{wall}}_{=0,\ \text{as}\ \boldsymbol{\chi}=0} - \int_\Omega p \nabla \cdot \boldsymbol{\chi} \, \mathrm{d}\Omega
\end{aligned}$$

Here, line integral over inlet is taken care of by Dirichlet BC, outlet integral vanishes as $p = 0$ there, symmetry line integral vanishes as $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ there, and finally no slip on the wall gets rid of the wall line integral.

$$\begin{aligned}
\int_\Omega \chi_i \frac{\partial}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\Omega &= \int_\Omega \frac{\partial}{\partial x_j} \left(\chi_i \frac{\partial v_i}{\partial x_j} \right) - \frac{\partial \chi_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\Omega \\
&= \int_{\partial\Omega} \chi_i \frac{\partial v_i}{\partial x_j} n_j \, \mathrm{d}\Gamma - \int_\Omega \frac{\partial \chi_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\Omega \\
&= \underbrace{\int_C \chi_i \frac{\partial v_i}{\partial x_j} n_j \, \mathrm{d}s_{sym}}_{=0,\ \text{as}\ \boldsymbol{n} \cdot \nabla \boldsymbol{v} = 0} - \int_\Omega \frac{\partial \chi_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\Omega
\end{aligned}$$

Here, line integrals over inlet boundary and wall boundary are taken care of by imposing Dirichlet BC on those boundaries, integrals over outlet and symmetry axis vanish due to natural boundary conditions

$$\begin{aligned}
\int_\Omega \chi_i \frac{\partial}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, \mathrm{d}\Omega &= \int_\Omega \frac{\partial}{\partial x_j} \left(\chi_i \frac{\partial v_j}{\partial x_i} \right) - \frac{\partial \chi_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, \mathrm{d}\Omega \\
&= \int_{\partial\Omega} \chi_i \frac{\partial v_j}{\partial x_i} n_j \, \mathrm{d}\Gamma - \int_\Omega \frac{\partial \chi_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, \mathrm{d}\Omega \\
&= \int_C \chi_i \frac{\partial v_j}{\partial x_i} n_j \, \mathrm{d}s_{sym} + \int_C \chi_i \frac{\partial v_j}{\partial x_i} n_j \, \mathrm{d}s_{outlet} - \int_\Omega \frac{\partial \chi_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, \mathrm{d}\Omega
\end{aligned}$$

here we also have Dirichlet BC on inlet, as well as a side wall, but outlet and symmetry boundary integrals do not vanish so need to include those terms in the formulation.

$$\begin{aligned}
\int_\Omega \chi_i \frac{\partial}{\partial x_j} \left((\nabla \cdot \boldsymbol{v}) \delta_{ij} \right) \mathrm{d}\Omega &= \int_\Omega \chi_i \frac{\partial}{\partial x_i} (\nabla \cdot \boldsymbol{v}) \, \mathrm{d}\Omega \\
&= \int_{\partial\Omega} (\nabla \cdot \boldsymbol{v}) \boldsymbol{\chi} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma - \int_\Omega (\nabla \cdot \boldsymbol{\chi})(\nabla \cdot \boldsymbol{v}) \, \mathrm{d}\Omega
\end{aligned}$$