Fluid Dynamics — Numerical Techniques

MATH5453M Numerical Exercise 3, 2016

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1 Ground water model

Consider the width-averaged nonlinear diffusion equation modelling groundwater flow in a channel (Barenblatt 1996)

$$\partial_t(w_v h_m) - \alpha g \partial_y(w_v h_m \partial_y h_m) = w_v R/(m_{por}\sigma_e) \tag{1}$$

with groundwater level and variable $h_m = h_m(y,t)$ $[L]$ above a horizontal datum $z = 0$, channel width $w_v \approx 0.1$m, acceleration of gravity $g = 9.81\text{m/s}^2$ $[L/T^2]$, derivative ∂_y in the along-channel direction $y \in [0, L_y]$ of a cell of length $L_y \approx 0.85$m $[L]$, porosity $m_{por} \in [0.1, 0.3]$, given rainfall $R = R(t)$ $[L/T]$, the fraction of a pore $\sigma_e \in [0.5, 1]$ that can be filled with water due to residual air, factor

$$\alpha = k/(\nu m_{por}\sigma_e) \tag{2}$$

with permeability $k \in [10^{-8}, 10^{-9}]\text{m}^2$ $[L^2]$ and viscosity of water $\nu = 10^{-6}\text{m}^2/\text{s}$ $[L^2/T]$. Boundary conditions are no flow at $y = L_y$ such that

$$\partial_y h_m = 0 \tag{3}$$

and a Dirichlet condition at $y = 0$, the other channel end, equal to the water level $h_{cm}(t)$ in a short outflow canal

$$h_m(0, t) = h_{cm}(t). \tag{4}$$

The initial condition is $h_m(y, 0) = h_{m0}(y)$. Make a sketch of the situation.

The assumptions are that the groundwater level stays underground thus not inducing any surface run-off and that the flow is hydrostatic, with variations in the horizontal y–direction much longer than the vertical length scales. The canal level $h_{cm}(t)$ holds in a short channel of length L_c between $-L_c < y < 0$ with a weir at $y = -L_c$. The water level at the weir is critical, meaning that the flow speed $V_c = \sqrt{gh_c}$ is critical at $y = -L_c$. Assuming stationarity and by using Bernoulli's equation to link the speed and water depth $\{V_{cm}, h_{cm}\}$ in the channel to that at the weir $\{V_c, h_c\}$ one obtains that

$$gh_{cm} + \frac{1}{2}V_{vm}^2 \approx gh_{cm} = gh_c + \frac{1}{2}V_c^2 = \frac{3}{2}gh_c, \tag{5}$$

$$\Longrightarrow h_c = \frac{2}{3}h_{cm} \quad \text{such that} \quad Q_c = h_c V_c = \sqrt{g}\max\left(2h_{cm}/3, 0\right)^{3/2}, \tag{6}$$

assuming in addition that $V_{cm}^2 \ll gh_{cm} \approx 0$ (see, e.g., Munson et al. 2005).

To obtain further insight, we rewrite and analyse (1) next. The groundwater equation is clearly a continuity equation

$$\partial_t(w_v h_m) + \partial_y(v h_m) = \frac{w_v R}{m_{por}\sigma_e} \tag{7}$$

with the Darcy velocity

$$v = -\frac{kg}{\nu m_{por}\sigma_e}\partial_y h_m \tag{8}$$

and Darcy flux

$$\begin{aligned} Q \equiv w_v q \equiv& w_v h_m v = -w_v h_m \frac{\kappa}{\mu}\partial_y p = -w_v h_m \frac{\kappa}{\nu\rho_0}\partial_y p \\ &\approx -\frac{\kappa g}{\nu}\partial_y h_m = -w_v h_m \frac{kg}{\nu m_{por}\sigma_e}\partial_y h_m = -w_v \alpha g \partial_y\left(h_m^2/2\right) \end{aligned} \tag{9}$$

with density of water ρ_0 and where we used the hydrostatic approximation and depth-integration to the free surface at $z = h_m$ by using $\partial_y p/\rho_0 \approx g\partial_y h_m$. The first and last term in (1) display the water balance as follows, in the case that there is no y–dependence:

$$\partial_t h_m = R/(m_{por}\sigma_e). \tag{10}$$

Hence, for h_m zero initially and constant rainfall, we find $h_m = tR/(m_{por}\sigma_e)$ showing that for $m_{por} = \sigma_e = 1$ unity the groundwater level rises directly with rainfall, while it rises faster for general $m_{por} < 1$ and $\sigma_e < 1$, showing that the modelling of rainfall supply is consistent. Hence, the canal level is modelled by the outflow at $y = -L_c$ and inflow at $y = 0$ as follows

$$L_c w_v \frac{\mathrm{d}h_{cm}}{\mathrm{d}t} = m_{por}\sigma_e Q_0 - Q_c \equiv w_v m_{por}\sigma_e \frac{1}{2}\alpha g \partial_y (h_m^2)|_{y=0} - w_v\sqrt{g}\max\left(\frac{2}{3}h_{cm}(t), 0\right)^{3/2}.$$

In summary, the complete mathematical groundwater model is:

$$\partial_t(w_v h_m) - \alpha g \partial_y(w_v h_m \partial_y h_m) = \frac{w_v R}{m_{por}\sigma_e} \quad \text{in} \quad y \in [0, L_y] \tag{11}$$

$$\partial_y h_m = 0 \quad \text{at} \quad y = L_y \tag{12}$$

$$h_m(0,t) = h_{cm}(t) \quad \text{at} \quad y = 0 \tag{13}$$

$$L_c w_v \frac{\mathrm{d}h_{cm}}{\mathrm{d}t} = w_v m_{por}\frac{\sigma_e}{2}\alpha g \partial_y (h_m^2)|_{y=0} - w_v\sqrt{g}\max\left(\frac{2}{3}h_{cm}(t), 0\right)^{3/2} \tag{14}$$

plus initial conditions for h_m and h_{cm}. The multiplication by $m_{por}\sigma$ accommodates the flow out of the groundwater matrix into open space (?), while the y–derivative of h_m^2 has been taken rather than using $h_m \partial_y h_m$ as otherwise a simple explicit discretisation with $h_m(0,0) = 0$ is and $h_{cm}(0) = 0$ will not lead to water flux into the canal. It may be necessary to rewrite the equations such that they fit an appropriate finite element weak formulation.

2 Questions

1. a) Write your own numerical program solving the heat equation $u_t = u_{xx}$ with Dirichlet boundary data as well as initial data using finite elements and using both the forward

Euler and Crank-Nicolson time stepping scheme. You can, e.g., use the Thomas algorithm or the backslash in matlab. Provide all steps required in the finite element discretisation in detail on paper before you start programming.

b) Argue how you can use the stability results for the finite difference scheme to find a time step for the finite element scheme. Van de Kan et al. (2005) provide more information on how to find the actual finite element stability criterion.

c) Compare the numerical results of the two implementations, for $\theta = 0, 1/2$. Use, for example, a top-hat profile with unit value in the middle and zero values at the edges, and $\exp(-\alpha(x - x_m)^2)$ for sufficiently large α and $x_m = 1/2$ lying within your domain. Compare your finite difference and finite element solutions. *Advanced: make a table of convergence using the* L^∞*–error.*

d) Numerically investigate the stability for $\theta = 0, 1/2$. Demonstrate this by succinctly showing your results in appropriate graphs. E.g., reproduce a relevant figure in Chapter 2 of Morton and Mayers (2005).

2. Discretise (11)–(14) using an explicit finite element space and time discretisation that keeps the adjoint structure in tact. Provide all steps required in detail first. Use an explicit scheme. Use the time step criterion for the finite difference case to obtain and state a time step estimate. Show how the flux at $y = 0$ arising after multiplication of the main partial differential equation by a test function and subsequent integration by parts can be eliminated. Should one take $h_m(0, t) = h_{cm}(t)$ in this calculation?

3. Perform numerical simulations for the parameter values:

$$m_{por} = 0.3, \quad \sigma_e = 0.8, \quad L_y = 0.85\text{m} \quad k = 10^{-8}\text{m}^2, \quad w_v = 0.1\text{m}, \tag{15}$$

$$R_{max} = 0.000125\text{m/s}, \quad L_c = 0.05\text{m}. \tag{16}$$

Start with $h_m(y, 0) = 0, h_{cm}(0) = 0$. Model the system for $t = 0, \dots, 100$s and give output profiles of $h_{cm}, R(t)$ every 2s and h_m every 10s. Demonstrate numerical convergence of your scheme for the solution at $t = 100$s. Has the system reached steady state; what is the steady-state value of h_{cm} and steady-state profile $h_m(y)$? Finally, vary the rain every 10s, apply rain $1, 2, 4$ or 9s out of 10s or at fixed lower percentages for fixed R_{max} and display the changes in $h_m(y,t), h_{cm}(t), R(t)$ sensibly. Compare with your finite difference solver and interpret your results. *Hint:* Perhaps first implement the easier problem with $h_{cm} = 0.07$m fixed and then add the canal equation.

4. *Advanced:* Use a Crank-Nicolson scheme instead to solve the above problem (in Firedrake). Detail your discretisation. Find a method to solve this nonlinear algebraic system (e.g., use Picard or Newton iteration). Implement it, demonstrate that the iteration converges and then compare it with the explicit (finite element and difference) discretisation.

References

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Internet, e.g., for table of convergence, L^∞–error, et cetera.

A Solutions

Given Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$, multiply

$$u_t = u_{xx} \tag{17}$$

with a test function $v = v(x)$ with $v(0) = v(L) = 0$ and integrate (by parts) to obtain

$$\int_0^L v u_t \, \mathrm{d}x = -\int_0^L v_x u_x \, \mathrm{d}x, \tag{18}$$

in which the boundary terms cancel because $v(0) = v(L) = 0$. This should be the weak formulation for Firedrake using v and u as a continuous Galerkin approximation with Dirichlet boundary conditions. For $L = 1$, the initial conditions could be, e.g., $u(x,0) = A\, e^{-\alpha|x-x_m|^2}$ with $x_m = L/2$ and suitable α such that $u(0,t) = u(L,t) \approx 0$; $u(x,0) = A\, x\,(1-x)$; or, $u(x,0) = A - \alpha(x-x_m)^2$ for $|x - x_m| \leq \sqrt{A/\alpha}$. Using compact (linear) basis functions $v = \varphi_{i'}(x)$ with $i' = 2, \ldots, N$ and Galerkin expansions

$$u(x,t) \approx u_h(x,t) = u_j(t)\varphi_j(x) = u_1\varphi_1 + u_{N+1}\varphi_{N+1} + u_{j'}\varphi_{j'} = u_{j'}\varphi_{j'}, \tag{19}$$

since $u_1 = u_{N+1} = 0$, we obtain

$$M_{i'j}\frac{\mathrm{d}u_j}{\mathrm{d}t} = -S_{i'j}u_j \Longrightarrow M_{i'j}\frac{\mathrm{d}u_{j'}}{\mathrm{d}t} = -S_{i'j'}u_{j'} \tag{20}$$

$$M_{i'j'} = \int_0^L \varphi_{i'}\varphi_{j'} \, \mathrm{d}x \quad \text{and} \quad S_{i'j'} = \int_0^L \partial_x\varphi_{i'}\partial_x\varphi_{j'} \, \mathrm{d}x, \tag{21}$$

with $(N-1)\times(N-1)$–matrices $M_{i'j'}$ and $S_{i'j'}$. Using the forward Euler and Crank-Nicolson time integration schemes, the weak Firedrake form and the matrix formulations become

$$\int_0^L v u^{n+1} \, \mathrm{d}x = \int_0^L v u^n - \Delta t v_x u_x^n \, \mathrm{d}x \tag{22}$$

$$\int_0^L v u^{n+1} + \frac{1}{2}\Delta t \int_0^L v_x u_x^{n+1} \, \mathrm{d}x = \int_0^L v u^n - \frac{1}{2}\Delta t \int_0^L v_x u_x^n \, \mathrm{d}x \tag{23}$$

$$M_{i'j'}{u_{j'}}^{n+1} = M_{i'j'}u_{j'}^n - \Delta t S_{i'j'}u_{j'}^n \tag{24}$$

$$\big(M_{i'j} + \frac{1}{2}\Delta t S_{i'j'}\big){u_{j'}}^{n+1} = \big(M_{i'j'} - \frac{1}{2}\Delta t S_{i'j'}\big)u_{j'}^n, \tag{25}$$

for which the initial condition needs to be projected onto the finite element basis using $\int_0^L v u_h(x,0)\,\mathrm{d}x = \int_0^L v u(x,0)\,\mathrm{d}x$ yielding $u_{j'}(0) = M^{-1}_{i'j'} \int_0^L \varphi_{i'} u(x,0)\,\mathrm{d}x$.

Groundwater model FEM: Multiplying (11) times test function $q = q(y)$ with $h_m(0,t) = h_{cm}(t)$ yields, after integration by parts and using that $\partial_y h_m = 0$ at $y = L_y$ as well as $h_m(0,t) = h_{cm}(t)$, that

$$\int_0^{L_y} q\partial_t h_m + \alpha g h_m \partial_y q \partial_y h_m \, dy + \underline{\alpha g q(0) h_m \partial_y h_m|_{y=0}} = \int_0^{L_y} \frac{qR}{m_{por}\sigma_e}\, dy \tag{26}$$

$$L_c \frac{\mathrm{d}h_{cm}}{\mathrm{d}t} = m_{por}\sigma_e \underline{\alpha g \frac{1}{2}\partial_y (h_m^2)|_{y=0}} - \sqrt{g}\max\left(\frac{2}{3}h_{cm}, 0\right)^{3/2}, \tag{27}$$

$$\underline{\alpha g \frac{1}{2}\partial_y (h_m^2)|_{y=0}} = L_c \frac{1}{m_{por}\sigma_e}\frac{\mathrm{d}h_{cm}}{\mathrm{d}t} + \frac{\sqrt{g}}{m_{por}\sigma_e}\max\left(\frac{2}{3}h_{cm}, 0\right)^{3/2}, \tag{28}$$

in which the underlined $\partial_y (h_m^2)|_{y=0}$ is eliminated between the two equations. These two equations are thus combined to obtain

$$\begin{aligned}\int_0^{L_y} q\partial_t h_m \, dy + \frac{q(0)L_c}{m_{por}\sigma_e}\frac{\mathrm{d}h_{cm}}{\mathrm{d}t} &= \int_0^{L_y} -\alpha g h_m \partial_y q \partial_y h_m + \frac{qR}{m_{por}\sigma_e}\, dy \\ &\quad - \frac{q(0)}{m_{por}\sigma_e}\sqrt{g}\max\left(\frac{2}{3}h_{cm}, 0\right)^{3/2}\end{aligned} \tag{29}$$

or

$$\begin{aligned}\int_0^{L_y} q\partial_t h_m \, dy + \frac{q(0)L_c}{m_{por}\sigma_e}\frac{\mathrm{d}h_m(0,t)}{\mathrm{d}t} &= \int_0^{L_y} -\alpha g h_m \partial_y q \partial_y h_m + \frac{qR}{m_{por}\sigma_e}\, dy \\ &\quad - \frac{q(0)}{m_{por}\sigma_e}\sqrt{g}\max\left(\frac{2}{3}h_m(0,t), 0\right)^{3/2}\end{aligned} \tag{30}$$

Note that q remains unconstraint. Consider piecewise linear finite elements. The forward

Euler and Crank-Nicolson time discretizations yield the (Firedrake) weak formulations

$$\int_0^{L_y} q h_m^{n+1}\,dy + \frac{L_c h_{cm}^{n+1}}{m_{por}\sigma_e} = \int_0^{L_y} q h_m^{n}\,dy + \frac{L_c h_{cm}^{n}}{m_{por}\sigma_e}$$
$$+\Delta t \int_0^{L_y} \left(-\alpha g h_m^n\,\partial_y q \partial_y h_m^n + \frac{qR^n}{m_{por}\sigma_e}\right) dy - \Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_{cm}^n, 0\right)^{3/2} \tag{31}$$

$$\int_0^{L_y} q h_m^{n+1}\,dy + \frac{L_c h_{cm}^{n+1}}{m_{por}\sigma_e} + \frac{1}{2}\Delta t \int_0^{L_y} \alpha g h_m^{n+1}\,\partial_y q \partial_y h_m^{n+1}\,dy + \frac{1}{2}\Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_{cm}^{n+1}, 0\right)^{3/2}$$
$$= \int_0^{L_y} q h_m^{n}\,dy + \frac{L_c h_{cm}^{n}}{m_{por}\sigma_e} + \frac{1}{2}\Delta t \int_0^{L_y} \left(-\alpha g h_m^n \partial_y q \partial_y h_m^n + \frac{q(R^n + R^{n+1})}{m_{por}\sigma_e}\right) dy$$
$$-\frac{1}{2}\Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_{cm}^n, 0\right)^{3/2}. \tag{32}$$

Taking expansions $q = \varphi_i(x)$ and $h_m = h_j \varphi_j(x)$, for all $i, j = 1, \dots, N_n$ with N_n nodes, as well as $h_1 = h_{cm}$ the matrix form for the forward Euler case becomes

$$M_{ij} h_j^{n+1} + \frac{L_c h_1^{n+1}}{m_{por}\sigma_e}\delta_{i1} = M_{ij} h_j^n + \frac{L_c h_1^n}{m_{por}\sigma_e}\delta_{i1} + \Delta t b_i^n - \Delta t \frac{\sqrt{g}}{m_{por}\sigma_e} \max\left(\frac{2}{3} h_1^n, 0\right)^{3/2} \delta_{i1}$$
$$\text{for} \quad i = 1, \dots, N_n \tag{33}$$

$$M_{ij} = \int_0^{L_y} \varphi_i \varphi_j\,dy \quad \text{and} \quad b_i^n = \int_0^{L_y} \left(-\alpha g h_m^n\,\partial_y \varphi_i \partial_y h_m^n + \frac{\varphi_i R^n}{m_{por}\sigma_e}\right) dy \tag{34}$$

with the Einstein summation convention used (here for j) and Kronecker delta symbol $\delta_{i1} = 1$ when $i = 1$ and $\delta_{i1} = 0$ when $i \neq 0$. Note that for a piecewise linear finite element approximation the unknown vector

$$(h_{cm}^{n+1}, h_2^{n+1}, \dots, h_j^{n+1}, \dots, h_{N_n}^{n+1})^T$$

includes the moor variables as well as the canal variable combined.

This formulation yields (visually) the same answer as the finite difference calculations!