Three-dimensional model of nonlinear potential flow equation using a change of coordinates

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1 Numerical implementation

1.1 Space discretization

To solve the weak formulations numerically, the surface and interior values of the potential ϕ need to be evaluated separately. For this, the discretization in the vertical coordinate σ is made by hand. We use the following Boussinesq-type discretization for the discrete potential $\phi_h(x, y, \sigma, t)$:

$$\phi_h(x, y, \sigma, t) = \psi_i(x, y, t)\tilde{\varphi}_i(\sigma). \tag{1}$$

The Einstein sum convention is used for repeated indices $i = [1, N_\sigma]$ where N_σ is the total number of nodes in the vertical. Substituting (1) into the variational principle yields

$$\begin{aligned} 0 = \delta \int_0^T \int_0^{L_x} \int_0^{L_y} \Bigg\{ \frac{1}{2}\frac{1}{H_0}\Bigg[h\nabla\psi_i \cdot \nabla\psi_j \tilde{M}_{ij} + \frac{1}{h}|\nabla h|^2 \psi_i\psi_j\tilde{S}_{ij} - 2\psi_i\nabla h \cdot \nabla\psi_j \tilde{D}_{ji} \Bigg] \\ + \frac{1}{2}\frac{H_0}{h}\psi_i\psi_j\tilde{A}_{ij} - \tilde{\varphi}_j(\sigma = H_0)\psi_j\partial_t h + \frac{1}{2}g(h - H_0)^2 \Bigg\} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2}$$

Where the matrices denote the σ-integrals as follow:

$$\tilde{M}_{ij} = \int_0^{H_0} \tilde{\varphi}_i\tilde{\varphi}_j \,\mathrm{d}\sigma \tag{3a}$$

$$\tilde{A}_{ij} = \int_0^{H_0} d_\sigma\tilde{\varphi}_i d_\sigma\tilde{\varphi}_j \,\mathrm{d}\sigma \tag{3b}$$

$$\tilde{D}_{ij} = \int_0^{H_0} \sigma\tilde{\varphi}_i d_\sigma\tilde{\varphi}_j \,\mathrm{d}\sigma \tag{3c}$$

$$\tilde{S}_{ij} = \int_0^{H_0} \sigma^2 d_\sigma\tilde{\varphi}_i d_\sigma\tilde{\varphi}_j \,\mathrm{d}\sigma \tag{3d}$$

One can notice that the time derivatives of h and ψ_j hold when j denotes the surface node only. Indeed,

$$\tilde{\varphi}_j(\sigma = H_0) = \begin{cases} 1 & \text{at} \quad \sigma_j = H_0 \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

If we denote by $j = 1$ the index for which $\tilde{\varphi}_j$ is evaluated at the surface, such that $\sigma_1 = H_0$, then from equation (2) the time derivative of h is non-zero only for $\tilde{\varphi}_1$. We therefore split the i and j indices into the surface nodes $(i, j) = (1, 1)$ and interior nodes $(i', j') = [2...N_z]$.

Then, the variational principle (2) becomes

$$
\begin{aligned}
0 = \delta \int_0^T \int_0^{L_{x,y}} \Bigg\{ & h \left|\nabla\psi_1\right|^2 \tilde{M}_{11} + \frac{1}{h}\left|\nabla h\right|^2 \psi_1^2 \tilde{S}_{11} - 2\psi_1 \nabla h \cdot \nabla\psi_1 \tilde{D}_{11} \\
& + \frac{H_0^2}{h}\psi_1^2 \tilde{A}_{11} - 2H_0 \tilde{\varphi}_1(H_0)\psi_1 \partial_t h + gH_0(h-H_0)^2 \\
& + \sum_{i'=2}^{N_z} \Bigg[2h\nabla\psi_{i'} \cdot \nabla\psi_1 \tilde{M}_{i'1} + \frac{2}{h}\left|\nabla h\right|^2 \psi_{i'}\psi_1 \tilde{S}_{i'1} - 2\psi_{i'}\nabla h \cdot \nabla\psi_1 \tilde{D}_{1i'} \\
& \qquad - 2\psi_1 \nabla h \cdot \nabla\psi_{i'} \tilde{D}_{i'1} + 2\frac{H_0^2}{h}\psi_{i'}\psi_1 \tilde{A}_{i'1} \\
& \qquad + \sum_{j'=2}^{N_z} \Bigg(h\nabla\psi_{i'} \cdot \nabla\psi_{j'} \tilde{M}_{i'j'} + \frac{1}{h}\left|\nabla h\right|^2 \psi_{i'}\psi_{j'} \tilde{S}_{i'j'} \\
& \qquad\qquad - 2\psi_{i'}\nabla h \cdot \nabla\psi_{j'} \tilde{D}_{j'i'} + \frac{H_0^2}{h}\psi_{i'}\psi_{j'}\tilde{A}_{i'j'} \Bigg) \Bigg] \Bigg\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}
\tag{5a}
$$

In the rest of this work, the implicit sum convention will be used for repeated indices. From there, Firedrake is used to solve the weak formulations in the (x,y,t) plane. For this, ψ_1, $\psi_{i'}$ and h are internally discretized as:

$$
\begin{aligned}
h(x,y,t) &= h_k(t)\varphi_k(x,y), \\
\psi_{i'}(x,y,t) &= \psi_{i'k}(t)\varphi_k(x,y), \\
\psi_1(x,y,t) &= \psi_{1k}(t)\varphi_k(x,y),
\end{aligned}
\tag{6}
$$

where $k = 1...N_s$ for N_s the total number of nodes in the horizontal plane. Applying the Stormer Verlet Scheme yields to the following system of weak formulations:

$$
\left\{
\begin{aligned}
&\int_0^{L_{x,y}} \varphi_k \psi_1^{n+1/2} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^{L_{x,y}} \Bigg\{ \varphi_k \psi_1^n - \frac{\Delta t}{4H_0} \Bigg[\varphi_k \left| \nabla \psi_1^{n+1/2} \right|^2 \tilde{M}_{11} - 2\psi_1^{n+1/2} \tilde{D}_{11} \left(\nabla \varphi_k \cdot \nabla \psi_1^{n+1/2} \right) \\
&\qquad + \tilde{S}_{11} \left(\psi_1^{n+1/2} \right)^2 \left(\frac{2}{h^n} \left(\nabla h^n \cdot \nabla \varphi_k \right) - \frac{\varphi_k}{(h^n)^2} \left| \nabla h^n \right|^2 \right) \\
&\qquad - \varphi_k \frac{H_0^2}{(h^n)^2} \left(\psi_1^{n+1/2} \right)^2 \tilde{A}_{11} + 2gH_0 \varphi_k \left(h^n - H_0 \right) \\
&\qquad + 2\varphi_k \left(\tilde{M}_{N1}^T \left(\nabla \hat{\boldsymbol{\psi}}^{n,+} \right)^T \nabla \psi_1^{n+1/2} \right) - 2 \left(\hat{\boldsymbol{\psi}}^{n,+} \tilde{D}_{1N}^T \right) \left(\nabla \varphi_k \cdot \nabla \psi_1^{n+1/2} \right) \\
&\qquad + 2\psi_1^{n+1/2} \left(\hat{\boldsymbol{\psi}}^{n,+} \tilde{S}_{N1} \right) \left(\frac{2}{h^n} \left(\nabla h^n \cdot \nabla \varphi_k \right) - \frac{\varphi_k}{(h^n)^2} \left| \nabla h^n \right|^2 \right) \\
&\qquad - 2\varphi_k \frac{H_0^2}{(h^n)^2} \left(\hat{\boldsymbol{\psi}}^{n,+} \tilde{A}_{N1} \right) \psi_1^{n+1/2} - 2\psi_1^{n+1/2} \left(\tilde{D}_{N1}^T \left(\nabla \hat{\boldsymbol{\psi}}^{n,+} \right)^T \nabla \varphi_k \right) \\
&\qquad + \varphi_k \left(\nabla \hat{\boldsymbol{\psi}}^{n,+} \tilde{M}_{NN} \right) \cdot \nabla \hat{\boldsymbol{\psi}}^{n,+} - 2 \left((\nabla \varphi_k)^T \nabla \hat{\boldsymbol{\psi}}^{n,+} \tilde{D}_{NN} (\hat{\boldsymbol{\psi}}^{n,+})^T \right) \\
&\qquad + \left(\hat{\boldsymbol{\psi}}^{n,+} \tilde{S}_{NN} \left(\hat{\boldsymbol{\psi}}^{n,+} \right)^T \right) \left(\frac{2}{h^n} \left(\nabla h^n \cdot \nabla \varphi_k \right) - \varphi_k \frac{\left| \nabla h^n \right|^2}{(h^n)^2} \right) \\
&\qquad - \varphi_k \frac{H_0^2}{(h^n)^2} \left(\hat{\boldsymbol{\psi}}^{n,+} \tilde{A}_{NN} (\hat{\boldsymbol{\psi}}^{n,+})^T \right) \Bigg] \Bigg\} \, \mathrm{d}x \, \mathrm{d}y, \\
&\text{For } i' \in [2, N+1] : \\
&\int_0^{L_{x,y}} \Bigg[\tilde{M}_{i'1} \left(\nabla \varphi_k \cdot \nabla \psi_1^{n+1/2} \right) h^n + \tilde{S}_{i'1} \psi_1^{n+1/2} \frac{\left| \nabla h^n \right|^2}{h^n} \varphi_k - \tilde{D}_{1i'} \left(\nabla h^n \cdot \nabla \psi_1^{n+1/2} \right) \varphi_k \\
&\qquad + \tilde{A}_{i'1} \psi_1^{n+1/2} \frac{H_0^2}{h^n} \varphi_k - \tilde{D}_{i'1} \psi_1^{n+1/2} \left(\nabla h^n \cdot \nabla \varphi_k \right) \Bigg] \, \mathrm{d}x \, \mathrm{d}y \\
&= - \int_0^{L_{x,y}} \Bigg[\left(\tilde{M}_{i'N} \left(\nabla \hat{\boldsymbol{\psi}}^{n,+} \right)^T \nabla \varphi_k \right) h^n + \left(\tilde{S}_{i'N} (\hat{\boldsymbol{\psi}}^{n,+})^T \right) \frac{\left| \nabla h^n \right|^2}{h^n} \varphi_k - \left(\tilde{D}_{Ni'}^T (\nabla \hat{\boldsymbol{\psi}}^{n,+})^T \nabla h^n \right) \varphi_k \\
&\qquad + \left(\tilde{A}_{i'N} (\hat{\boldsymbol{\psi}}^{n,+})^T \right) \frac{H_0^2}{h^n} \varphi_k - \left(\tilde{D}_{i'N} (\hat{\boldsymbol{\psi}}^{n,+})^T \right) \left(\nabla h^n \cdot \nabla \varphi_k \right) \Bigg] \, \mathrm{d}x \, \mathrm{d}y,
\end{aligned}
\right.
\tag{7}
$$

$$
\left\{
\begin{aligned}
&\int_0^{L_{x,y}} \varphi_m h^{n+1}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{L_{x,y}} \Bigg\{ \varphi_m h^n + \frac{\Delta t}{2H_0}\Bigg[\tilde{M}_{11}\nabla\psi_1^{n+1/2}\cdot\nabla\varphi_m\left(h^n+h^{n+1}\right) \\
&\qquad + \tilde{S}_{11}\varphi_m\psi_1^{n+1/2}\left(\frac{|\nabla h^n|^2}{h^n}+\frac{|\nabla h^{n+1}|^2}{h^{n+1}}\right) \\
&\qquad - \tilde{D}_{11}\varphi_m\left(\nabla\psi_1^{n+1/2}\cdot\nabla h^n+\nabla\psi_1^{n+1/2}\cdot\nabla h^{n+1}\right) \\
&\qquad - \tilde{D}_{11}\psi_1^{n+1/2}\left(\nabla\varphi_m\cdot\nabla h^n+\nabla\varphi_m\cdot\nabla h^{n+1}\right) \\
&\qquad + H_0^2\tilde{A}_{11}\psi_1^{n+1/2}\varphi_m\left(\frac{1}{h^n}+\frac{1}{h^{n+1}}\right) \\
&\qquad + h^n\left(\tilde{M}_{N1}^T(\nabla\hat{\boldsymbol{\psi}}^{n,+})^T\nabla\varphi_m\right) + h^{n+1}\left(\tilde{M}_{N1}^T(\nabla\hat{\boldsymbol{\psi}}^{n+1})^T\nabla\varphi_m\right) \\
&\qquad + \varphi_m\left(\frac{|\nabla h^n|^2}{h^n}\left(\hat{\boldsymbol{\psi}}^{n,+}\tilde{S}_{N1}\right)+\frac{|\nabla h^{n+1}|^2}{h^{n+1}}\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{S}_{N1}\right)\right) \\
&\qquad - \tilde{D}_{1N}\Big((\hat{\boldsymbol{\psi}}^{n,+})^T\left(\nabla h^n\cdot\nabla\varphi_m\right)+(\hat{\boldsymbol{\psi}}^{n+1})^T\left(\nabla h^{n+1}\cdot\nabla\varphi_m\right)\Big) \\
&\qquad - \varphi_m\left(\tilde{D}_{N1}^T(\nabla\hat{\boldsymbol{\psi}}^{n,+})^T\nabla h^n+\tilde{D}_{N1}^T(\nabla\hat{\boldsymbol{\psi}}^{n+1})^T\nabla h^{n+1}\right) \\
&\qquad + H_0^2\varphi_m\left(\frac{1}{h^n}\left(\hat{\boldsymbol{\psi}}^{n,+}\tilde{A}_{N1}\right)+\frac{1}{h^{n+1}}\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{A}_{N1}\right)\right)\Bigg]\Bigg\}\,\mathrm{d}x\,\mathrm{d}y, \\
&\text{For } i'\in[2,N_z]: \\
&\int_0^{L_{x,y}}\Bigg[\tilde{M}_{1i'}\left(\nabla\psi_1^{n+1/2}\cdot\nabla\varphi_m\right)h^{n+1}+\psi_1^{n+1/2}\tilde{S}_{1i'}\varphi_m\frac{|\nabla h^{n+1}|^2}{h^{n+1}}-\tilde{D}_{1i'}\left(\nabla\psi_1^{n+1/2}\cdot\nabla h^{n+1}\right)\varphi_m \\
&\qquad -\tilde{D}_{i'1}\psi_1^{n+1/2}\left(\nabla h^{n+1}\cdot\nabla\varphi_m\right)+\psi_1^{n+1/2}\tilde{A}_{1i'}\frac{H_0^2}{h^{n+1}}\varphi_m\Bigg]\,\mathrm{d}x\,\mathrm{d}y \\
&= -\int_0^{L_{x,y}}\Bigg[\left(\tilde{M}_{Ni'}^T(\nabla\hat{\boldsymbol{\psi}}^{n+1})^T\nabla\varphi_m\right)h^{n+1}+\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{S}_{Ni'}\right)\varphi_m\frac{|\nabla h^{n+1}|^2}{h^{n+1}}-\left(\tilde{D}_{Ni'}^T(\nabla\hat{\boldsymbol{\psi}}^{n+1})^T\nabla h^{n+1}\right)\varphi_m \\
&\qquad -\big(\tilde{D}_{i'N}(\hat{\boldsymbol{\psi}}^{n+1})^T\big)\big(\nabla h^{n+1}\cdot\nabla\varphi_m\big)+\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{A}_{Ni'}\right)\frac{H_0^2}{h^{n+1}}\varphi_m\Bigg]\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
\right.
\tag{8}
$$

$$
\begin{aligned}
&\int_0^{L_{x,y}} \varphi_k \psi_1^{n+1}\,\mathrm{d}x\,\mathrm{d}y \\
= &\int_0^{L_{x,y}} \Bigg\{ \varphi_k \psi_1^{n+1/2} - \frac{\Delta t}{4H_0} \Bigg[\varphi_k \left|\nabla \psi_1^{n+1/2}\right|^2 \tilde{M}_{11} - 2\psi_1^{n+1/2}\left(\nabla\psi_1^{n+1/2}\cdot\nabla\varphi_k\right)\tilde{D}_{11} \\
&\quad + \tilde{S}_{11}(\psi_1^{n+1/2})^2\left(\frac{2}{h^{n+1}}\left(\nabla h^{n+1}\cdot\nabla\varphi_k\right) - \frac{\varphi_k}{(h^{n+1})^2}\left|\nabla h^{n+1}\right|^2\right) \\
&\quad - \varphi_k \frac{H_0^2}{(h^{n+1})^2}(\psi_1^{n+1/2})^2\tilde{A}_{11} + 2gH_0\varphi_k\left(h^{n+1}-H_0\right) \\
&\quad + 2\varphi_k\left(\tilde{M}_{N1}^T(\nabla\hat{\boldsymbol{\psi}}^{n+1})^T\nabla\psi_1^{n+1/2}\right) - 2\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{D}_{1N}^T\right)\left(\nabla\psi_1^{n+1/2}\cdot\nabla\varphi_k\right) \\
&\quad + 2\psi_1^{n+1/2}\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{S}_{N1}\right)\left(\frac{2}{h^{n+1}}\left(\nabla h^{n+1}\cdot\nabla\varphi_k\right) - \frac{\varphi_k}{(h^{n+1})^2}|\nabla h^{n+1}|^2\right) \\
&\quad - 2\varphi_k\frac{H_0^2}{(h^{n+1})^2}\psi_1^{n+1/2}\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{A}_{N1}\right) - 2\psi_1^{n+1/2}\left(\tilde{D}_{N1}^T\left(\nabla\hat{\boldsymbol{\psi}}^{n+1}\right)^T\nabla\varphi_k\right) \\
&\quad + \varphi_k\left(\nabla\hat{\boldsymbol{\psi}}^{n+1}\tilde{M}_{NN}\right).\nabla\hat{\boldsymbol{\psi}}^{n+1} - 2\left((\nabla\varphi_k)^T\nabla\hat{\boldsymbol{\psi}}^{n+1}\tilde{D}_{NN}(\hat{\boldsymbol{\psi}}^{n+1})^T\right) \\
&\quad\; + \left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{S}_{NN}\left(\hat{\boldsymbol{\psi}}^{n+1}\right)^T\right)\left(\frac{2}{h^{n+1}}\left(\nabla h^{n+1}\cdot\nabla\varphi_k\right) - \varphi_k\frac{|\nabla h^{n+1}|^2}{(h^{n+1})^2}\right) \\
&\quad\; - \varphi_k\frac{H_0^2}{(h^{n+1})^2}\left(\hat{\boldsymbol{\psi}}^{n+1}\tilde{A}_{NN}(\hat{\boldsymbol{\psi}}^{n+1})^T\right)\Bigg]\Bigg\}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
\tag{9}
$$

Where the three unknown functions are:

- $h(x, y, t)$, a function defined in the horizontal domain
- $\psi_1(x, y, t)$, a function defined in the horizontal domain
- $\hat{\boldsymbol{\psi}}$, a vector function containing all the $\psi_{i'}$ functions, that is $\hat{\boldsymbol{\psi}} = [\psi_2, \psi_3, ..., \psi_{N_z}]$ for a N^{th} order expansion in the vertical (*i.e.* $N_z = N + 1$ nodes), where each of the component is a function defined in the horizontal plane.

and the σ-discretized matrices are split them into four sub-matrices, constant in space and time:

$$
\begin{aligned}
X_{11} &= X_{ij}[1,1], \\
X_{1N} &= X_{1j'} = X_{ij}[1, 2:N_z], \\
X_{N1} &= X_{i'1} = X_{ij}[2:N_z, 1], \\
X_{NN} &= X_{i'j'} = X_{ij}[2:N_z, 2:N_z],
\end{aligned}
\tag{10}
$$

]where X denotes any matrix among $\tilde{A}$, $\tilde{B}$, $\tilde{M}$ and $\tilde{S}$. Figure 1, illustrates the definition of these four submatrices in the case of the mass matrix $\tilde{M}$.

1.2 Evaluation of the vertical integrals

While the horizontal discretization is made internally, the matrices (3) must be evaluated analitically in order to be used as coefficients in the horizontal weak formulations (**??**)-(**??**)-(9). The domain in the σ-direction is discretized with only one element, on which we extend the σ-dependent basis function $\tilde{\varphi}_i(\sigma)$ with a high order Lagrange polynomial, that is

$$
\tilde{\varphi}_i(\sigma) = \prod_{\substack{k=1 \\ k\neq i}}^{N_z} \frac{\sigma - \sigma_k}{\sigma_i - \sigma_k},
\tag{11}
$$

$$\hat{M}_{ij} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1(N+1)} \\ M_{21} & M_{22} & \cdots & M_{2(N+1)} \\ \vdots & & \ddots & \vdots \\ M_{(N+1)1} & \cdots & & M_{(N+1)(N+1)} \end{pmatrix} = \begin{pmatrix} \hat{M}_{11} & \hat{M}_{1N} \\ \hat{M}_{N1} & \hat{M}_{NN} \end{pmatrix}$$

Figure 1: The vertically-discretised matrices are split into 4 time and space constant sub-matrices.

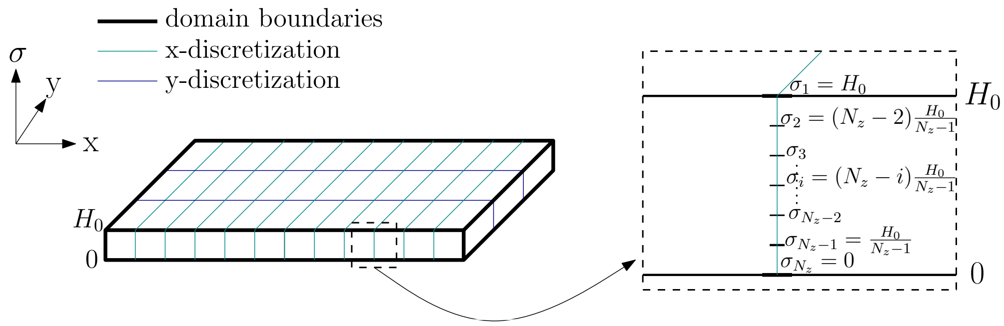

Figure 2: Discretization of the domain in the $\sigma-$direction. The basis function $\tilde{\varphi}_i(\sigma)$ is expanded using a Lagrange polynomial of high order n in one element only.

where $N_z = n+1$ is the number of evaluation nodes for an expansion of order n. The discrete coordinate σ_k is defined as

$$\sigma_k = (N_z - k)\frac{H_0}{N_z - 1}, \tag{12}$$

for a depth of water H_0, so that $\sigma_1 = H_0$ denotes the surface coordinate and $\sigma_{N_z} = 0$ denotes the bottom of the domain. A scheme of this discretization is available on Figure 2.