The following three equations need to be solved simultaneously for $\phi^{n+1/2}$, η^{n+1} and $\lambda^{n+1/2}$. The weak formulation for Firedrake has the following form and includes the use of a Heaviside step function $\Theta(x - L_p)$ and three test functions $v_{1,2,3}$,

$$\int_0^L v_1\Big(\phi^{n+1/2} - \phi^n + \frac{\Delta t}{2} g\eta^n - \Theta(x - L_p)\lambda^{n+1/2}\Big)\,\mathrm{d}x = 0 \tag{1a}$$

$$\int_0^L \Big(v_2(\eta^{n+1} - \eta^n) - \Delta t H(x)\partial_x v_2 \partial_x \phi^{n+1/2}\Big)\,\mathrm{d}x = 0 \tag{1b}$$

$$\int_0^L v_3\Theta(x - L_p)\Big(\frac{1}{\Delta t}(\eta^{n+1} - Z^n) - W^n + \frac{\rho}{m}\int_0^L \Theta(\tilde{x} - L_p)\lambda^{n+1/2}(\tilde{x})\,\mathrm{d}\tilde{x}\Big)\,\mathrm{d}x = 0. \tag{1c}$$

On attempting to solve to problem as a mixed system using Schur complements, the tricky part is how to implement tha last equation (1c). In finite element matrix form with $\lambda = \lambda_\ell \varphi_\ell$ etc (under summation convention), this will become

$$\frac{1}{\Delta t}(M_{k\ell}\eta^{n+1} - Q_k Z^n) - Q_k W^n + \frac{\rho}{m} Q_k Q_\ell \lambda_\ell^{n+1/2} = 0, \tag{2}$$

with modified mass matrix $M_{k\ell}$ (includes also the heavyside function) and vector Q_k.

Algorithm using real function spaces:

1. Define the following integral, which is a constant:

$$I = \int_0^L \Theta(\tilde{x} - L_p)\lambda^{n+1/2}(\tilde{x})\,\mathrm{d}\tilde{x}. \tag{3a}$$

2. Multiply by test function v_4, which is a constant in the real space, and integrate in $[0, L]$:

$$\int_0^L v_4\Big(I - \int_0^L \Theta(\tilde{x} - L_p)\lambda^{n+1/2}(\tilde{x})\,\mathrm{d}\tilde{x}\Big)\,\mathrm{d}x = 0. \tag{3b}$$

3. The second term on the RHS is independent of x so it can go outside the integral, which will then be equal to the domain length L:

$$\int_0^L v_4 I\,\mathrm{d}x - v_4 L\int_0^L \Theta(\tilde{x} - L_p)\lambda^{n+1/2}(\tilde{x})\,\mathrm{d}\tilde{x} = 0. \tag{3c}$$

4. Take v_4 inside the integral, since it is a constant, and replace dummy variable $\tilde{x}$ by x:

$$\int_0^L v_4\Big(\frac{I}{L} - v_4\Theta(x - L_p)\lambda^{n+1/2}(x)\,\mathrm{d}x\Big) = 0. \tag{3d}$$

Equation (3d) can be now solved in conjuction with (1a)-(1c) and, by replacing the red term in (1c) by I, a solution can be sought for the 4 variables $\phi^{n+1/2}$, η^{n+1} and $\lambda^{n+1/2}$ and I.