The aim is to solve the following weak formulations simultaneously

$$
\left\{
\begin{aligned}
&\int_0^{L_{x,y}} \varphi_k \psi_1^{n+1/2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{L_{x,y}} \Bigg\{ \varphi_k\psi_1^n - \frac{\Delta t}{4H_0}\Bigg[\varphi_k|\nabla\psi_1^{n+1/2}|^2\tilde{M}_{11} + \frac{2}{h^n}\nabla h^n\cdot\nabla\varphi_k(\psi_1^{n+1/2})^2\tilde{S}_{11} \\
&\qquad - \frac{\varphi_k}{(h^n)^2}|\nabla h^n|^2\left(\psi_1^{n+1/2}\right)^2\tilde{S}_{11} - 2\psi_1^{n+1/2}\nabla\varphi_k\cdot\nabla\psi_1^{n+1/2}\tilde{D}_{11} \\
&\qquad - \frac{H_0^2}{(h^n)^2}\varphi_k\left(\psi_1^{n+1/2}\right)^2\tilde{A}_{11} + 2gH_0\varphi_k\left(h^n - H_0\right) \\
&\qquad + 2\varphi_k\nabla(\hat{\boldsymbol{\psi}}^{n,+}\tilde{M}_{N1})\cdot\nabla\psi_1^{n+1/2} \\
&\qquad - 2\frac{\varphi_k}{(h^n)^2}|\nabla h^n|^2\left(\hat{\boldsymbol{\psi}}^{n,+}\tilde{S}_{N1}\right)\psi_1^{n+1/2} \\
&\qquad + \frac{4}{h^n}\nabla h^n\cdot\nabla\varphi_k\left(\hat{\boldsymbol{\psi}}^{n,+}\tilde{S}_{N1}\right)\psi_1^{n+1/2} \\
&\qquad - 2\left((\hat{\boldsymbol{\psi}}^{n,+1})^T\tilde{D}_{1N}\right)\nabla\varphi_k\cdot\nabla\psi_1^{n+1/2} \\
&\qquad - 2\frac{H_0^2}{(h^n)^2}\varphi_k\left(\hat{\boldsymbol{\psi}}^{n,+}\tilde{A}_{N1}\right)\psi_1^{n+1/2} \\
&\qquad - 2\psi_1^{n+1/2}\nabla\varphi_k\cdot\left(\nabla\hat{\boldsymbol{\psi}}^{n,+}\tilde{D}_{N1}\right) \\
&\qquad + \varphi_k\nabla\hat{\boldsymbol{\psi}}^{n,+}\tilde{M}_{NN}\left(\nabla\hat{\boldsymbol{\psi}}^{n,+}\right)^T - \varphi_k\frac{|\nabla h^n|^2}{(h^n)^2}\hat{\boldsymbol{\psi}}^{n,+}\tilde{S}_{NN}\left(\hat{\boldsymbol{\psi}}^{n,+}\right)^T \\
&\qquad - 2\nabla\varphi_k\cdot\left(\nabla\hat{\boldsymbol{\psi}}^{n,+}\tilde{D}_{NN}(\hat{\boldsymbol{\psi}}^{n,+})^T\right) \\
&\qquad + \frac{2}{h^n}\nabla h^n\cdot\nabla\varphi_k\hat{\boldsymbol{\psi}}^{n,+}\tilde{S}_{NN}(\hat{\boldsymbol{\psi}}^{n,+})^T \\
&\qquad - \frac{H_0^2}{(h^n)^2}\varphi_k\hat{\boldsymbol{\psi}}^{n,+}\tilde{A}_{NN}(\hat{\boldsymbol{\psi}}^{n,+})^T\Bigg]\Bigg\}\,\mathrm{d}x\,\mathrm{d}y, \\
&\qquad\text{For } i' \in [2, N+1]: \\
&\int_0^{L_{x,y}} \Bigg[2h^n\nabla\varphi_k\cdot\nabla\psi_1^{n+1/2}\tilde{M}_{i'1} + \frac{2}{h^n}\left|\nabla h^n\right|^2\varphi_k\psi_1^{n+1/2}\tilde{S}_{i'1} - 2\varphi_k\nabla h^n\cdot\nabla\psi_1^{n+1/2}\tilde{D}_{1i'} \\
&\qquad +2\frac{H_0^2}{h^n}\varphi_k\psi_1^{n+1/2}\tilde{A}_{i'1} - 2\psi_1^{n+1/2}\nabla h^n\cdot\nabla\varphi_k\tilde{D}_{i'1}\Bigg]\,\mathrm{d}x\,\mathrm{d}y \\
&= -\int_0^{L_{x,y}} \Bigg[2h^n\nabla\varphi_k\cdot\left(\tilde{M}_{i'N}(\nabla\hat{\boldsymbol{\psi}}^{n,+})^T\right) + \frac{2}{h^n}\left|\nabla h^n\right|^2\varphi_k\tilde{S}_{i'N}(\hat{\boldsymbol{\psi}}^{n,+})^T - 2\varphi_k\nabla h^n\cdot\left(\nabla\hat{\boldsymbol{\psi}}^{n,+}\tilde{D}_{Ni'}\right) \\
&\qquad +2\frac{H_0^2}{h^n}\varphi_k\tilde{A}_{i'N}(\hat{\boldsymbol{\psi}}^{n,+})^T - 2\tilde{D}_{i'N}(\hat{\boldsymbol{\psi}}^{n,+})^T\nabla h^n\cdot\nabla\varphi_k\Bigg]\,\mathrm{d}x\,\mathrm{d},
\end{aligned}
\right.
\tag{1}
$$

for the unknowns $\psi_1^{n+1/2}$ and $\hat{\boldsymbol{\psi}}^{n,+}$, where :

- φ_k is a test function
- h is a function defined in the horizontal domain
- ψ_1 is a function defined in the horizontal domain
- $\hat{\boldsymbol{\psi}}$ is a vector containing N columns, each of them being a function defined in the horizontal plane, that is $\hat{\boldsymbol{\psi}} = [\psi_2, \psi_3, ..., \psi_{N+1}]$, where ψ_i, for $i \in [2, ..., N+1]$ is defined in the horizontal domain.

Therefore, this is a system of $(N+1)$ equations for $(N+1)$ unknowns : $\psi_1^{n+1/2}$ and $\hat{\boldsymbol{\psi}}^{n,+} = [\psi_2^{n,+}, ..., \psi_{N+1}^{n,+}]$.

The functions h, ψ_1 and $\hat{\boldsymbol{\psi}}$ are defined on Firedrake as follow:

```
mesh = RectangleMesh(Nx,Ny,Lx,Ly)
V = FunctionSpace(mesh, "CG", 1)
Vec = VectorFunctionSpace(mesh,"CG",1,dim=n)
W = V*Vec

h = Function(V)
w1 = Function(W)
psi_1, hat_psi = split(w1)
v, q = TestFunction(W)
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However, some matrices, constant in space and time, are also involved in the weak formulations. They represent the integral over z of some products of Lagrange polynomials and their derivatives. For instance,

$$M_{ij} = \int_0^{H_0} \varphi_i \varphi_j \, \mathrm{d}z, \tag{2}$$

where φ_i is a Lagrange polynomial of n^{th} order defined as

$$\varphi_i = \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{z - z_k}{z_i - z_k}, \tag{3}$$

where z_k denotes the discrete z coordinate. These matrices are split into four sub-matrices, defined that way:

$$\hat{M}_{ij} = \begin{pmatrix} M_{11} & M_{12} & \dots & & M_{1(N+1)} \\ M_{21} & M_{22} & & \dots & M_{2(N+1)} \\ \vdots & & \ddots & & \vdots \\ M_{(N+1)1} & & \dots & & M_{(N+1)(N+1)} \end{pmatrix} = \begin{pmatrix} \hat{M}_{11} & \hat{M}_{1N} \\ \hat{M}_{N1} & \hat{M}_{NN} \end{pmatrix}$$

where the size of $\hat{M}_{11}$, $\hat{M}_{1N}$, $\hat{M}_{N1}$ and $\hat{M}_{NN}$ are $(1,1)$, $(1,N)$, $(N,1)$ and (N,N) respectively.

Then, the question is: Since these matrices are not defined via Firedrake, but as some arrays, how can we compute the products such as $\hat{\boldsymbol{\psi}}^{n,+} M_{n,1}$ involved in the weak formulations?