

The following three equations need to be solved simultaneously for $\phi^{n+1/2}$, η^{n+1} and $\lambda^{n+1/2}$. The weak formulation for Firedrake has the following form and includes the use of a Heaviside step function $\Theta(x - L_p)$ and three test functions $v_{1,2,3}$,

$$\int_0^L v_1 \left(\phi^{n+1/2} - \phi^n + \frac{\Delta t}{2} g \eta^n - \Theta(x - L_p) \lambda^{n+1/2} \right) dx = 0 \quad (1a)$$

$$\int_0^L \left(v_2 (\eta^{n+1} - \eta^n) - \Delta t H(x) \partial_x v_2 \partial_x \phi^{n+1/2} \right) dx = 0 \quad (1b)$$

$$\int_0^L v_3 \Theta(\tilde{x} - L_p) \left(\frac{1}{\Delta t} (\eta^{n+1} - Z^n) - W^n + \frac{\rho}{m} \int_0^L \Theta(x - L_p) \lambda^{n+1/2} dx \right) d\tilde{x} = 0. \quad (1c)$$

On attempting to solve to problem as a mixed system using Schur complements, the tricky part is how to implement the last equation (1c). In finite element matrix form with $\lambda = \lambda_\ell \varphi_\ell$ etc (under summation convention), this will become

$$\frac{1}{\Delta t} (M_{k\ell} \eta^{n+1} - Q_k Z^n) - Q_k W^n + \frac{\rho}{m} Q_k Q_\ell \lambda_\ell^{n+1/2} = 0, \quad (2)$$

with modified mass matrix $M_{k\ell}$ (includes also the heavyside function) and vector Q_k .