Motion of a buoy in linear shallow water

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In what follows, $\phi(x,t)$ is the velocity potential, $\eta(x,t)$ the free surface deviation, $Z(t)$ the position of the buoy's centre of mass and $W(t)$ its speed. ρ is the water's density, g is gravity and M is the buoy's mass, which are both constants. λ is a Lagrange multiplier used to impose the constraint $\eta_b = Z$ under the buoy.

1 Firedrake formulation

The weak formulation for Firedrake has the following form and includes the use of a Heaviside function $\Theta(x-L_p)$, zero for $x < l_p$ and one for $x \geq L_p$, as follows:

$$\int_0^L \delta\eta_h \phi_h^{n+1/2}\,\mathrm{d}x = \int_0^L \delta\eta_h \big(\phi_h^n - \frac{\Delta t}{2} g\eta_h^n + \frac{\Delta t}{2}\Theta(x-L_p)\,\delta\eta_h\lambda_h^{n+1/2}\big)\,\mathrm{d}x \quad (1a)$$

$$MW^{n+1/2} = MW^n - \frac{\Delta t}{2}\rho\int_0^L \Theta(x-L_p)\,\lambda_h^{n+1/2}\,\mathrm{d}x \quad (1b)$$

$$\int_0^L \delta\phi_h\eta_h^{n+1}\,\mathrm{d}x = \int_0^L \delta\phi_h\eta_h^n\,\mathrm{d}x \quad (1c)$$

$$+\int_0^L \big(H_0\Theta(x-L_p) + H_b\Theta(L_p-x)\big)\big(\Delta t\partial_x\delta\phi_h\partial_x\phi_h^{n+1/2}\big)\,\mathrm{d}x$$

$$Z^{n+1} = Z^n + \Delta t W^{n+1/2} \quad (1d)$$

$$0 = \int_0^L \delta\lambda_h\,\big(\Theta(L_p-x)\,\lambda_h^{n+1/2} + \Theta(x-L_p)(\eta_h^{n+1} - Z^{n+1})\big)\,\mathrm{d}x \quad (1e)$$

$$= \int_0^L \delta\lambda_h\,\Big(\Theta(L_p-x)\,\lambda_h^{n+1/2} + \Theta(x-L_p)(\eta_h^{n+1} - Z^n - \Delta t W^n)$$

$$+\,\Theta(x-L_p)\frac{(\Delta t)^2}{2}\frac{\rho}{M}\int_0^L \Theta(x-L_p)\lambda_h^{n+1/2}\,\mathrm{d}x\Big)\,\mathrm{d}x \quad (1f)$$

Equations (1a)-(1e) have to be solved simultaneously for $\phi_h^{n+1/2}$, $W^{n+1/2}$, $\lambda_h^{n+1/2}$, η_h^{n+1}, Z^{n+1} (in matrix form it can be solved for $\lambda_h^{n+1/2}$ first and then the rest of the steps are explicit, see section 2). Alternatively, one can eliminate Z^{n+1} and $W^{n+1/2}$ and solve for the remaining three unknowns, i.e. use eq. (1f) instead.

The final two steps are explicit anf are given below

$$\int_0^L \delta\eta_h \phi_h^{n+1}\,\mathrm{d}x = \int_0^L \delta\eta_h \big(\phi_h^{n+1/2} - \frac{\Delta t}{2} g\eta_h^{n+1} + \frac{\Delta t}{2}\Theta(x-L_p)\,\lambda_h^{n+1/2}\big)\,\mathrm{d}x \tag{1g}$$

$$MW^{n+1} = MW^{n+1/2} - \frac{\Delta t}{2}\rho\int_0^L \Theta(x-L_p)\rho\lambda_h^{n+1/2}\,\mathrm{d}x. \tag{1h}$$

2 Discretisation

Consider a C^0–finite element spatial discretization. All nodes are denoted by $\hat{i},\hat{j}$. Nodes at or under the free water surface are denoted by i,j with the subset of nodes at the free surface denoted by k,l. Nodes under the hull are denoted by $\tilde{i}$ and the subset of nodes at the hull's surface by $\tilde{k},\tilde{l}$. Hence, the finite element expansions used are (the Einstein summation convention is adopted)

$$\phi_h(x,z,t) = \phi_{\hat{i}}(t)\varphi_{\hat{i}}(x,z),\quad \eta_h(x,t) = \ \eta_l(t)\varphi_l(x)\quad \text{or}\quad \eta_h(x,t) = \eta_{\hat{l}}(t)\varphi_{\hat{l}}(x).$$

Defining the following matrices

$$M_{\hat{k}\hat{l}} = \int_0^L \varphi_{\hat{k}}\varphi_{\hat{l}}\,\mathrm{d}x, \qquad Q_{\tilde{l}} = \int_{L_p}^L \varphi_{\tilde{l}}|_{H_b(x,\bar{Z})}\,\mathrm{d}x, \tag{2}$$
$$A_{\hat{i}\hat{j}} = \int_0^{L_p} H_0\partial_x\varphi_{\hat{i}}\cdot\partial_x\varphi_{\hat{j}}\,\mathrm{d}x + \int_{L_p}^L H_b(x,\bar{Z})\partial_x\varphi_{\hat{i}}\cdot\partial_x\varphi_{\hat{j}}\,\mathrm{d}x.$$

the equations of motion are

$$\begin{aligned}
\delta\lambda_{\tilde{k}}:&\quad M_{\tilde{k}\tilde{l}}\eta_{\tilde{l}} - Q_{\tilde{k}}Z = 0\\
\delta\eta_{\hat{k}}:&\quad M_{\hat{k}\hat{l}}\dot{\phi}_{\hat{l}} + gM_{\hat{k}\hat{l}}\eta_{\hat{l}} - \delta_{\hat{k}\tilde{l}}M_{\tilde{m}\tilde{l}}\lambda_{\tilde{m}} = 0\\
\delta\phi_{\hat{l}}:&\quad M_{\hat{k}\hat{l}}\dot{\eta}_{\hat{k}} - A_{\hat{k}\hat{l}}\phi_{\hat{k}} = 0\\
\delta Z:&\quad M\dot{W} + \rho Q_{\tilde{k}}\lambda_{\tilde{k}} = 0\\
\delta W:&\quad \dot{Z} = W.
\end{aligned} \tag{3}$$

Such a constraint problem can be discretized in time using various versions of the RATLLE algorithm, the constraint extension of the symplectic second order Störmer-Verlet scheme for constraint systems [1, 2, 3, 4]. We then obtain the following time discrete version of (3)

$$M_{\hat{k}\hat{l}}\phi_{\hat{l}}^{n+1/2} = M_{\hat{k}\hat{l}}\phi_{\hat{l}}^n - \frac{\Delta t}{2}gM_{\hat{k}\hat{l}}\eta_{\hat{l}}^n + \frac{\Delta t}{2}\delta_{\hat{k}\tilde{l}}M_{\tilde{m}\tilde{l}}\lambda_{\tilde{m}}^{n+1/2} \tag{4a}$$

$$MW^{n+1/2} = MW^n - \frac{\Delta t}{2}\rho Q_{\tilde{k}}\lambda_{\tilde{k}}^{n+1/2} \tag{4b}$$

$$M_{\hat{k}\hat{l}}\eta_{\hat{k}}^{n+1} = M_{\hat{k}\hat{l}}\eta_{\hat{k}}^n + \Delta t A_{\hat{k}\hat{l}}\phi_{\hat{k}}^{n+1/2} \tag{4c}$$

$$Z^{n+1} = Z^n + \Delta t W^{n+1/2} \tag{4d}$$

$$0 = M_{\tilde{k}\tilde{l}}\eta_{\tilde{l}}^{n+1} - Q_{\tilde{k}}Z^{n+1} \tag{4e}$$

$$M_{\hat{k}\hat{l}}\phi_{\hat{l}}^{n+1} = M_{\hat{k}\hat{l}}\phi_{\hat{l}}^{n+1/2} - \frac{\Delta t}{2}gM_{\hat{k}\hat{l}}\eta_{\hat{l}}^{n+1} + \frac{\Delta t}{2}\delta_{\hat{k}\tilde{l}}M_{\tilde{m}\tilde{l}}\lambda_{\tilde{m}}^{n+1/2} \tag{4f}$$

$$MW^{n+1} = MW^{n+1/2} - \frac{\Delta t}{2}\rho Q_{\tilde{k}}\lambda_{\tilde{k}}^{n+1/2} \tag{4g}$$

The Lagrange multiplier $\lambda^{n+1/2}$ is determined by substituting the updates $\eta_{\tilde{l}}^{n+1}$ and Z^{n+1} into the discrete constraint (4e), leading to a linear equation for this Lagrange multiplier.

References

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