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Firedrake: towards generating an extruded dynamical core

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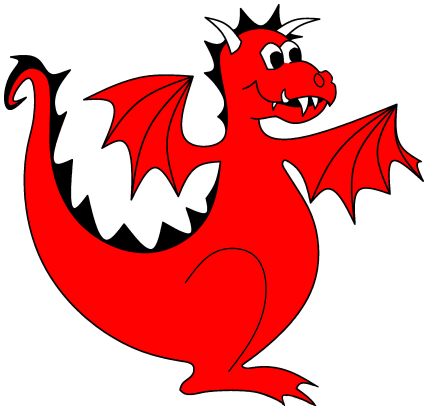
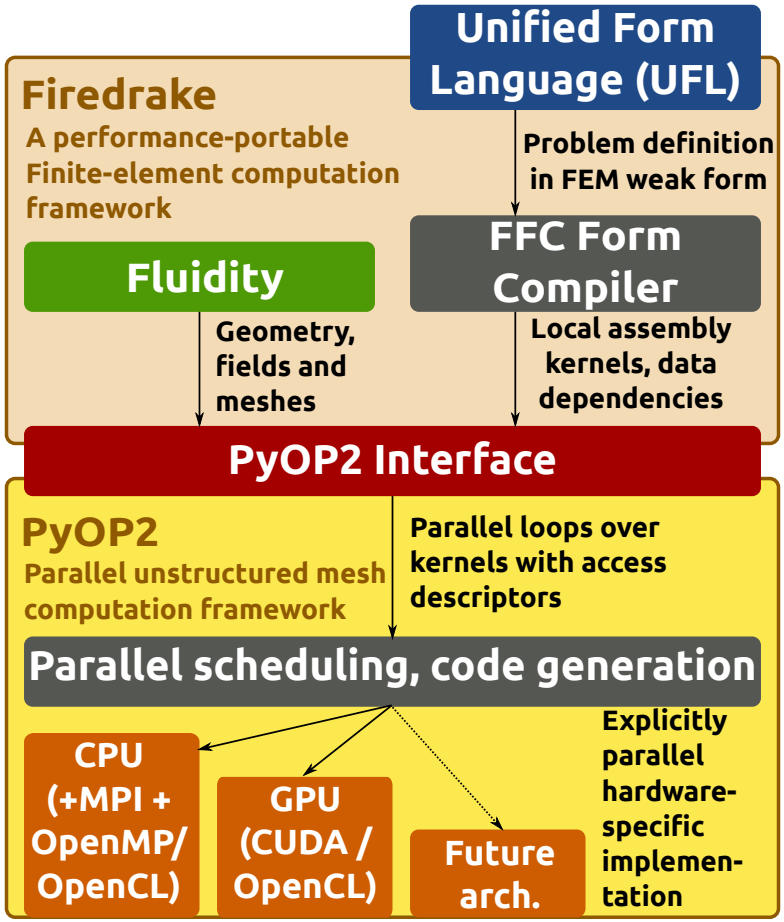


Firedrake vs GungHo/LFRic

- Same data model.
- Essentially the same execution model.
- Firedrake generates code at runtime, GungHo is compile time.
- Firedrake generates code for discretisations, GungHo currently does not plan to.
- Firedrake currently only supports triangles in the horizontal. Primal quads on roadmap, primal-dual is harder.

The Firedrake system



The model problem

$$-\nabla^2 p = f \tag{1}$$

on $\Omega = [0,1] \times [0,1] \times [0,1]$ with:

$$p = 0 \quad \text{on } \Gamma_D = \{x = 0, x = 1, y = 0, y = 1\} \tag{2}$$

$$\nabla p \cdot n = g \quad \text{on } \Gamma_N = \{z = 0, z = 1\} \tag{3}$$

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In dual form:

$$u - \nabla p = 0 \tag{4}$$

$$-\nabla \cdot u = f \tag{5}$$

Model problem in weak form

$$u - \nabla p = 0 \tag{4}$$
$$-\nabla \cdot u = f \tag{5}$$

In weak form, find $(u, p) \in W_2' \times W_3$ such that:

$$\int_\Omega u \cdot v + p \nabla \cdot v \mathrm{d}x + \cancel{\int_{\Gamma_D} p \mathbf{n} \cdot v \mathrm{d}s} = 0 \quad \forall \mathbf{v} \in W_2' \tag{6}$$
$$\int_\Omega -(\nabla \cdot u) q \mathrm{d}x = \int_\Omega f q \mathrm{d}x \quad \forall q \in W_3 \tag{7}$$

where $W_2' = \{u \in W_2, u \cdot n = g \text{ on } \Gamma_N\}$.

Building a mesh

```
from firedrake import *

m = UnitSquareMesh(4, 4)
mesh = ExtrudedMesh(m, 11, layer_height=0.1)
```

It's also possible to specify a more complex expression for layer heights.

Building the function space

$$W_2 = H_{\mathrm{Div}}\left(\mathrm{BDM}_1 \times \mathrm{DG}_0 + \mathrm{DG}_1 \times \mathrm{CG}_1\right) \tag{8}$$

```
W2_a_horiz = FiniteElement("BDM", "triangle", 1)
W2_a_vert = FiniteElement("DG", "interval", 0)
W2_a = HDiv(OuterProductElement(W2_a_horiz,
                                W2_a_vert))

W2_b_horiz = FiniteElement("DG", "triangle", 1)
W2_b_vert = FiniteElement("CG", "interval", 1)
W2_b = HDiv(OuterProductElement(W2_b_horiz,
                                W2_b_vert))

W2_elt = W2_a + W2_b
W2 = FunctionSpace(mesh, W2_elt)
```

Building the function space

$$W_3 = DG_0 \times DG_0 W = W_2 \times W_3 \tag{9}$$

```
W3 = FunctionSpace(mesh, "DG", 0, vfamily="DG",
                                  vdegree=0)

return W2 * W3
```


Alternative function space

```
# Note the Norwegian counting.
def RT2():
    W2_a_horiz = FiniteElement("RT", "triangle", 2)
    W2_a_vert = FiniteElement("DG", "interval", 1)
    W2_a = HDiv(OuterProductElement(W2_a_horiz,
                                    W2_a_vert))

    W2_b_horiz = FiniteElement("DG", "triangle", 1)
    W2_b_vert = FiniteElement("CG", "interval", 2)
    W2_b = HDiv(OuterProductElement(W2_b_horiz,
                                    W2_b_vert))

    W2_elt = W2_a + W2_b
    W2 = FunctionSpace(mesh, W2_elt)

    W3 = FunctionSpace(mesh, "DG", 1, vfamily="DG",
                                      vdegree=1)

    return W2 * W3
```

Choose a right hand side

$$f = 3\pi^2 \sin(\pi x)\sin(\pi y)\sin(\pi z) \tag{10}$$

yielding solution

$$p = \sin(\pi x)\sin(\pi y)\sin(\pi z) \tag{11}$$

```
f = Function(W3)
exact = Function(W3)
f.interpolate(Expression("(3*pi*pi)*sin(x[0]*pi)*sin(x
                          [1]*pi)*sin(x[2]*pi)"))
exact.interpolate(Expression("sin(x[0]*pi)*sin(x[1]*pi
                          )*sin(x[2]*pi)"))
```

Set matching boundary conditions

$$u \cdot z = \pi \sin(\pi x) \sin(\pi y) \text{ on } z = 0 \tag{12}$$

$$u \cdot z = -\pi \sin(\pi x) \sin(\pi y) \text{ on } z = 1 \tag{13}$$

```
bcs = [DirichletBC(W[0], Expression(("0.0", "0.0", "pi
                                     *sin(x[0]*pi)*sin(x[1]*pi)"
                                     )), "bottom"),
       DirichletBC(W[0], Expression(("0.0", "0.0", "-
                                           pi*sin(x[0]*pi)*sin(
                                           x[1]*pi)")), "top")]
```

Now actually specify the variational problem

$$\int_\Omega u \cdot v + p\nabla \cdot v \mathrm{d}x = 0 \quad \forall \mathbf{v} \in W_2' \tag{14}$$

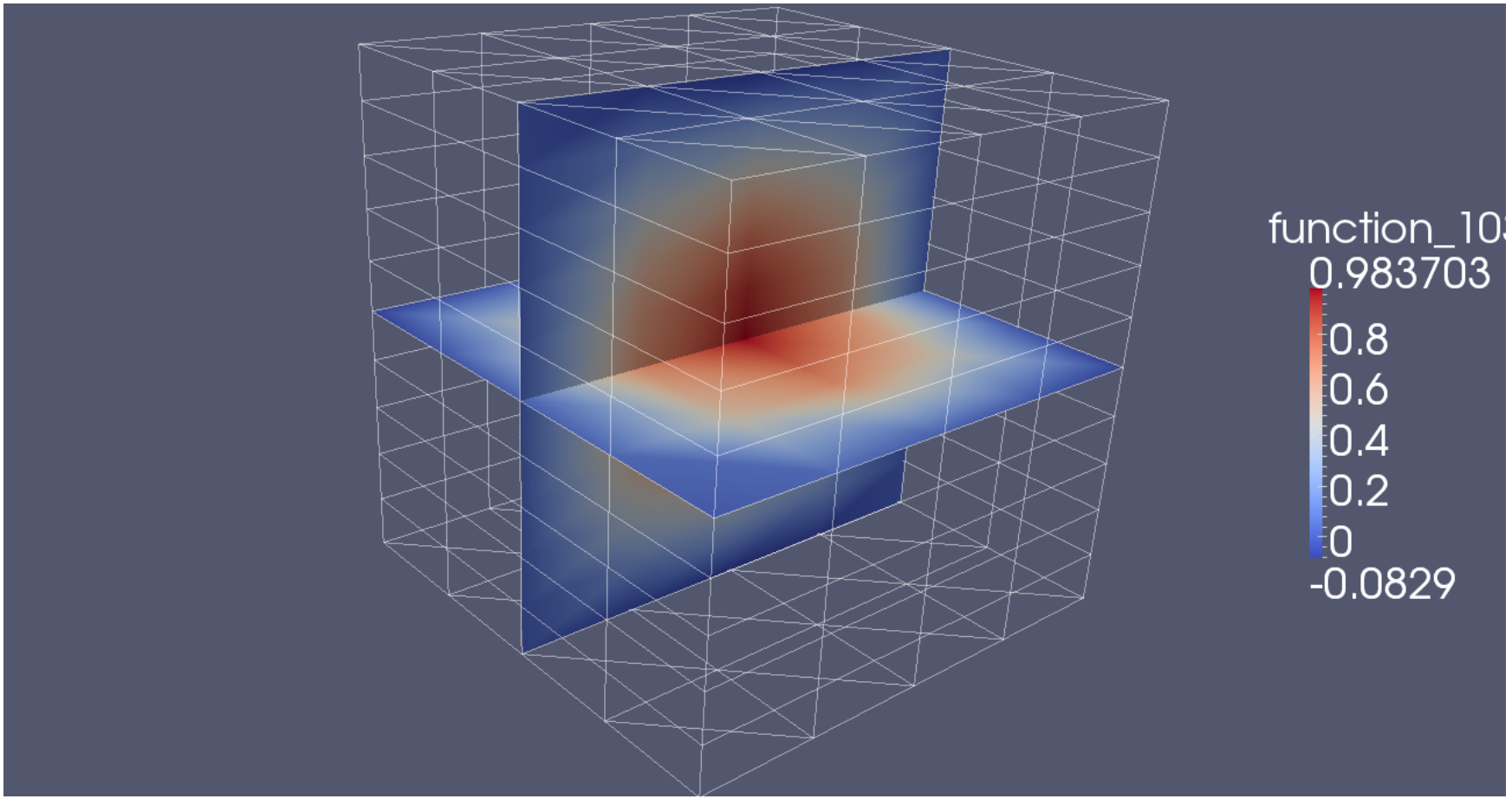
$$\int_\Omega -(\nabla \cdot u) q \mathrm{d}x = \int_\Omega f q \mathrm{d}x \quad \forall q \in W_3 \tag{15}$$

```
u, p = TrialFunctions(W)
v, q = TestFunctions(W)
a = (-q*div(u) + dot(v, u) + div(v)*p)*dx
L = f*q*dx
```


Define a solution function, and solve

```
out = Function(W)
solve(a == L, out, bcs=bcs, solver_parameters={
      'pc_type': 'fieldsplit',
      'pc_fieldsplit_type': 'schur',
      'ksp_type': 'cg',
      'pc_fieldsplit_schur_fact_type': 'FULL',
      'fieldsplit_0_ksp_type': 'cg',
      'fieldsplit_1_ksp_type': 'cg'})
```


The final result



Development roadmap

- Vertical slice (just bugs away)
- Lateral BCs (including periodic)
- Facet integrals (interior and exterior)
- Hybridisation
- Replace the mesh data structures with DMPLEX from PETSc
- Quads
- Somewhat less stupid tensor product evaluation
- Multigrid
- General performance engineering

Finally

http://firedrakeproject.org

http://wp.doc.ic.ac.uk/spo/finite-element/